Geometry 1: Manifolds and sheaves

Rules: You may choose to solve only "hard" exercises (marked with !, * and **) or "ordinary" ones (marked with ! or unmarked), or both, if you want to have extra problems. To have a perfect score, a student must obtain (in average) a score of 10 points per week. It's up to you to ignore handouts entirely, because passing tests in class and having good scores at final exams could compensate (at least, partially) for the points obtained by grading handouts.

Solutions for the problems are to be explained to the examiners orally in the class and marked in the score sheet. It's better to have a written version of your solution with you. It's OK to share your solutions with other students, and use books, Google search and Wikipedia, we encourage it. The first score sheet will be distributed February 11-th.

If you have got credit for 2/3 of ordinary problems or 2/3 of "hard" problems, you receive $6t$ points, where t is a number depending on the date when it is done. Passing all "hard" or all "ordinary" problems (except at most 2) brings you $10t$ points. Solving of "**" (extra hard) problems is not obligatory, but each such problem gives you a credit for 2 "*" or "!" problems in the "hard" set.

The first 3 weeks after giving a handout, $t = 1.5$, between 21 and 35 days, $t = 1$, and afterwards, $t = 0.7$. The scores are not cumulative, only the best score for each handout counts.

Please keep your score sheets until the final evaluation is given.

The original English translation of this handout was done by Sasha Anan′in (UNICAMP) in 2010.

1.1 Topological manifolds

Remark 1.1. Manifolds can be smooth (of a given "class of smoothness"), real analytic, or topological (continuous). **Topological manifold** is easiest to define, it is a topological space which is locally homeomorphic to an open ball in $\mathbb{R}^n$.

Definition 1.1. **An action** of a group on a manifold is silently assumed to be continuous. Let G be a group acting on a set M. The **stabilizer** of $x \in M$ is the subgroup of all elements in G that fix x. An action is **free** if the stabilizer of every point is trivial.

Exercise 1.1. Let G be a finite group acting freely on a Hausdorff manifold M. Show that the quotient space M/G is a manifold.

Exercise 1.2. Construct an example of a finite group G acting non-freely on a manifold M such that M/G is not a manifold.

Exercise 1.3. Consider the quotient of $\mathbb{R}^2$ by the action of $\{\pm 1\}$ that maps x to $-x$. Is the quotient space a manifold?

Exercise 1.4 (*). Let M be a path connected, Hausdorff topological manifold, and G a group of all its homeomorphisms. Prove that G acts on M transitively.

Exercise 1.5 (**). Prove that any closed subgroup $G \subset GL(n)$ of a matrix group is homeomorphic to a manifold, or find a counterexample.

Remark 1.2. In the above definition of a manifold, it is not required to be Hausdorff. Nevertheless, in most cases, manifolds are silently assumed to be Hausdorff.

Exercise 1.6. Construct an example of a non-Hausdorff manifold.

Exercise 1.7. Show that $\mathbb{R}^2/\mathbb{Z}^2$ is a manifold.

Exercise 1.8. Let α be an irrational number. The group $\mathbb{Z}^2$ acts on $\mathbb{R}$ by the formula $t \mapsto t + m + n\alpha$. Show that this action is free, but the quotient $\mathbb{R}/\mathbb{Z}^2$ is not a manifold.

Exercise 1.9 (**). Construct an example of a (non-Hausdorff) manifold of positive dimension such that the closures of two arbitrary nonempty open sets always intersect, or show that such a manifold does not exist.

Exercise 1.10 (**). Let $G \subset GL(n, \mathbb{R})$ be a compact subgroup. Show that the quotient space $GL(n, \mathbb{R})/G$ is also a manifold.

1.2 Smooth manifolds

Definition 1.2. A **cover** of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$. A cover $\{V_i\}$ is a **refinement** of a cover $\{U_i\}$ if every V_i is contained in some U_i.

Exercise 1.11. Show that any two covers of a topological space admit a common refinement.

Definition 1.3. A cover $\{U_i\}$ is an **atlas** if for every U_i, we have a map $\varphi_i : U_i \to \mathbb{R}^n$ giving a homeomorphism of U_i with an open subset in $\mathbb{R}^n$. The **transition maps**

$$\Phi_{ij} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

are induced by the above homeomorphisms. An atlas is **smooth** if all transition maps are smooth (of class C^∞, i.e., infinitely differentiable), **smooth of class** C^i if all transition functions are of differentiability class C^i, and **real analytic** if all transition maps admit a Taylor expansion at each point.

Definition 1.4. A **refinement** of an **atlas** is a refinement of the corresponding cover $V_i \subset U_i$ equipped with the maps $\varphi_i : V_i \to \mathbb{R}^n$ that are the restrictions of $\varphi_i : U_i \to \mathbb{R}^n$. Two atlases (U_i, φ_i) and (U_i, ψ_i) of class C^∞ or C^i (with the same cover) are **equivalent** in this class if, for all i, the map $\psi_i \circ \varphi_i^{-1}$ defined on the corresponding open subset in $\mathbb{R}^n$ belongs to the mentioned class. Two arbitrary atlases are **equivalent** if the corresponding covers possess a common refinement giving equivalent atlases.

Definition 1.5. A **smooth structure** on a manifold (of class C^∞ or C^i) is an atlas of class C^∞ or C^i considered up to the above equivalence. A **smooth manifold** is a topological manifold equipped with a smooth structure.

Remark 1.3. Terrible, isn't it?

Exercise 1.12 (*). Construct an example of two nonequivalent smooth structures on $\mathbb{R}^n$.

Definition 1.6. A **smooth function** on a manifold M is a function f whose restriction to the chart (U_i, φ_i) gives a smooth function $f \circ \varphi_i^{-1}: \varphi_i(U_i) \longrightarrow \mathbb{R}$ for each open subset $\varphi_i(U_i) \subset \mathbb{R}^n$.

Remark 1.4. There are several ways to define a smooth manifold. The above way is most standard. It is not the most convenient one but you should know it. Two other ways (via sheaves of functions and via Whitney's theorem) are presented further in these handouts.

Definition 1.7. A **presheaf of functions** on a topological space M is a collection of subrings $\mathcal{F}(U) \subset C(U)$ in the ring $C(U)$ of all functions on U, for each open subset $U \subset M$, such that the restriction of every $\gamma \in \mathcal{F}(U)$ to an open subset $U_1 \subset U$ belongs to $\mathcal{F}(U_1)$.

Definition 1.8. A presheaf of functions $\mathcal{F}$ is called **a sheaf of functions** if these subrings satisfy the following condition. Let $\{U_i\}$ be a cover of an open subset $U \subset M$ (possibly infinite) and $f_i \in \mathcal{F}(U_i)$ a family of functions defined on the open sets of the cover and compatible on the pairwise intersections:

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

for every pair of members of the cover. Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.

Remark 1.5. **A presheaf of functions** is a collection of subrings of functions on open subsets, compatible with restrictions. **A sheaf of fuctions** is a presheaf allowing "gluing" a function on a bigger open set if its restriction to smaller open sets lies in the presheaf.

Definition 1.9. A sequence $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow ...$ of homomorphisms of abelian groups or vector spaces is called **exact** if the image of each map is the kernel of the next one.

Exercise 1.13. Let $\mathcal{F}$ be a presheaf of functions. Show that $\mathcal{F}$ is a sheaf if and only if for every cover $\{U_i\}$ of an open subset $U \subset M$, the sequence of restriction maps

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$$

is exact, with $\eta \in \mathcal{F}(U_i)$ mapped to $\eta\Big|_{U_i \cap U_j}$ and $-\eta\Big|_{U_j \cap U_i}$.

Exercise 1.14. Show that the following spaces of functions on $\mathbb{R}^n$ define sheaves of functions.

a. Space of continuous functions.

b. Space of smooth functions.

c. Space of functions of differentiability class C^i.

d. (*) Space of functions that are pointwise limits of sequences of continuous functions.

e. Space of functions vanishing outside a set of measure 0.

Exercise 1.15. Show that the following spaces of functions on $\mathbb{R}^n$ are presheaves, but not sheaves

a. Space of constant functions.

b. Space of bounded functions.

c. Space of functions vanishing outside of a bounded set.

d. Space of continuous functions with finite $\int |f|$.

Definition 1.10. A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of functions. A **morphism** $(M, \mathcal{F}) \xrightarrow{\Psi} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \xrightarrow{\Psi} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $f \circ \Psi$ belongs to the ring $\mathcal{F}\big(\Psi^{-1}(U)\big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

Remark 1.6. Usually the term "ringed space" stands for a more general concept, where the "sheaf of functions" is an abstract "sheaf of rings," not necessarily a subsheaf in the sheaf of all functions on M. The above definition is simpler, although not as standard.

Exercise 1.16. Let M, N be open subsets in $\mathbb{R}^n$ and let $\Psi : M \to N$ be a smooth map. Show that Ψ defines a morphism of spaces ringed by smooth functions.

Exercise 1.17. Let M be a smooth manifold of some class and let $\mathcal{F}$ be the space of functions of this class. Show that $\mathcal{F}$ is a sheaf.

Exercise 1.18 (!). Let M be a topological manifold, and let (U_i, φ_i) and (V_j, ψ_j) be smooth structures on M. Show that these structures are equivalent if and only if the corresponding sheaves of smooth functions coincide.

Remark 1.7. This exercise implies that the following definition is equivalent to the one stated earlier.

Definition 1.11. Let $(M, \mathcal{F})$ be a topological manifold equipped with a sheaf of functions. It is said to be a **smooth manifold** of **class** C^∞ or C^i if every point in $(M, \mathcal{F})$ has an open neighborhood isomorphic to the ringed space $(\mathbb{R}^n, \mathcal{F}')$, where $\mathcal{F}'$ is a ring of functions on $\mathbb{R}^n$ of this class.

Definition 1.12. A **coordinate system** on an open subset U of a manifold $(M, \mathcal{F})$ is an isomorphism between $(U, \mathcal{F})$ and an open subset in $(\mathbb{R}^n, \mathcal{F}')$, where $\mathcal{F}'$ are functions of the same class on $\mathbb{R}^n$.

Remark 1.8. In order to avoid complicated notation, from now on we assume that all manifolds are Hausdorff and smooth (of class C^∞). The case of other differentiability classes can be considered in the same manner.

Exercise 1.19 (!). Let $(M, \mathcal{F})$ and $(N, \mathcal{F}')$ be manifolds and let $\Psi : M \to N$ be a continuous map. Show that the following conditions are equivalent.
(i) In local coordinates, Ψ is given by a smooth map
(ii) Ψ is a morphism of ringed spaces.

Remark 1.9. An isomorphism of smooth manifolds is called a **diffeomorphism**. As follows from this exercise, a diffeomorphism is a homeomorphism that maps smooth functions onto smooth ones.

Exercise 1.20 (*). Let $\mathcal{F}$ be a presheaf of functions on $\mathbb{R}^n$. Figure out a minimal sheaf that contains $\mathcal{F}$ in the following cases.

a. Constant functions.

b. Functions vanishing outside a bounded subset.

c. Bounded functions.

Exercise 1.21 (*). Describe all morphisms of ringed spaces from $(\mathbb{R}^n, C^{i+1})$ to $(\mathbb{R}^n, C^i)$.

1.3 Embedded manifolds

Definition 1.13. A **closed embedding** $\phi : N \hookrightarrow M$ of topological spaces is an injective map from N to a closed subset $\phi(N)$ inducing a homeomorphism of N and $\phi(N)$. **An open embedding** $\phi : N \hookrightarrow M$ is a homeomorphism of N and an open subset of M. is an image of a closed embedding.

Definition 1.14. Let M be a smooth manifold. $N \subset M$ is called **smoothly embedded submanifold of dimension** m if for every point $x \in N$, there is a neighborhood $U \subset M$ diffeomorphic to an open ball $B \subset \mathbb{R}^n$, such that this diffeomorphism maps $U \cap N$ onto a linear subspace of B dimension m.

Exercise 1.22. Let $(M, \mathcal{F})$ be a smooth manifold and let $N \subset M$ be a smoothly embedded submanifold. Consider the space $\mathcal{F}'(U)$ of smooth functions on $U \subset N$ that are extendable to functions on M defined on some neighborhood of U.

a. Show that $\mathcal{F}'$ is a sheaf.

b. Show that this sheaf defines a smooth structure on N.

c. Show that the natural embedding $(N, \mathcal{F}') \to (M, \mathcal{F})$ is a morphism of manifolds.

Hint. To prove that $\mathcal{F}$ is a sheaf, you might need partition of unity introduced below. Sorry.

Exercise 1.23. Let N_1, N_2 be two manifolds and let $\varphi_i : N_i \to M$ be smooth embeddings. Suppose that the image of N_1 coincides with that of N_2. Show that N_1 and N_2 are isomorphic.

Remark 1.10. By the above problem, in order to define a smooth structure on N, it suffices to embed N into $\mathbb{R}^n$. As it will be clear in the next handout, every manifold is embeddable into $\mathbb{R}^n$ (assuming it admits partition of unity). Therefore, in place of a smooth manifold, we can use "manifolds that are smoothly embedded into $\mathbb{R}^n$."

Exercise 1.24. Construct a smooth embedding of $\mathbb{R}^2/\mathbb{Z}^2$ into $\mathbb{R}^3$.

Exercise 1.25 (**). Show that the projective space $\mathbb{R}P^n$ does not admit a smooth embedding into $\mathbb{R}^{n+1}$ for $n > 1$.

1.4 Partition of unity

Exercise 1.26. Show that an open ball $\mathbb{B}^n \subset \mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$.

Definition 1.15. A cover $\{U_\alpha\}$ of a topological space M is called **locally finite** if every point in M possesses a neighborhood that intersects only a finite number of U_α.

Exercise 1.27. Let $\{U_\alpha\}$ be a locally finite atlas on M, and $U_\alpha \xrightarrow{\phi_\alpha} \mathbb{R}^n$ homeomorphisms. Consider a cover $\{V_i\}$ of $\mathbb{R}^n$ given by open balls of radius n centered in integer points, and let $\{W_\beta\}$ be a cover of M obtained as union of $\phi_\alpha^{-1}(V_i)$. Show that $\{W_\beta\}$ is locally finite.

Exercise 1.28. Let $\{U_\alpha\}$ be an atlas on a manifold M.

a. Construct a refinement $\{V_\beta\}$ of $\{U_\alpha\}$ such that a closure of each V_β is compact in M.

b. Prove that such a refinement can be chosen locally finite if $\{U_\alpha\}$ is locally finite

Hint. Use the previous exercise.

Exercise 1.29. Let K_1, K_2 be non-intersecting compact subsets of a Hausdorff topological space M. Show that there exist a pair of open subsets $U_1 \supset K_1$, $U_2 \supset K_2$ satisfying $U_1 \cap U_2 = \emptyset$.

Exercise 1.30 (!). Let $U \subset M$ be an open subset with compact closure, and $V \supset M \backslash U$ another open subset. Prove that there exists $U' \subset U$ such that the closure of U' is contained in U, and $V \cup U' = M$.

Hint. Use the previous exercise

Definition 1.16. Let $U \subset V$ be two open subsets of M such that the closure of U is contained in V. In this case we write $U \Subset V$.

Exercise 1.31 (!). Let $\{U_\alpha\}$ be a countable locally finite cover of a Hausdorff topological space, such that a closure of each U_α is compact. Prove that there exists another cover $\{V_\alpha\}$ indexed by the same set, such that $V_\alpha \Subset U_\alpha$

Hint. Use induction and the previous exercise.

Exercise 1.32 (*). Solve the previous exercise when $\{U_\alpha\}$ is not necessarily countable.

Hint. Some form of transfinite induction is required.

Exercise 1.33 (!). Denote by $\mathbb{B} \subset \mathbb{R}^n$ an open ball of radius 1. Let $\{U_i\}$ be a locally finite countable atlas on a manifold M. Prove that there exists a refinement $\{V_i, \phi_i : \ V_i \tilde{\longrightarrow} \mathbb{R}^n\}$ of $\{U_i\}$ which is also locally finite, and such that $\bigcup_i \phi_i^{-1}(\mathbb{B}) = M$.

Hint. Use Exercise 1.31 and Exercise 1.28.

Definition 1.17. A function with compact support is a function which vanishes outside of a compact set.

Definition 1.18. Let M be a smooth manifold and let $\{U_\alpha\}$ be a locally finite cover of M. A **partition of unity** subordinate to the cover $\{U_\alpha\}$ is a family of smooth functions $f_i : M \to [0,1]$ with compact support indexed by the same indices as the U_i's and satisfying the following conditions.

(a) Every function f_i vanishes outside U_i

(b) $\sum_i f_i = 1$

Remark 1.11. Note that the sum $\sum_i f_i = 1$ makes sense only when $\{U_\alpha\}$ is locally finite.

Exercise 1.34. Show that all derivatives of $e^{-\frac{1}{x^2}}$ at 0 vanish.

Exercise 1.35. Define the following function λ on $\mathbb{R}^n$

$$\lambda(x) := \begin{cases} e^{\frac{1}{|x|^2-1}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

Show that λ is smooth and that all its derivatives vanish at the points of the unit sphere.

Exercise 1.36. Let $\{U_i, \ \varphi_i : U_i \tilde{\longrightarrow} \mathbb{R}^n\}$ be an atlas on a smooth manifold M. Consider the following function $\lambda_i : M \to [0,1]$

$$\lambda_i(m) := \begin{cases} \lambda\big(\varphi_i(m)\big) & \text{if } m \in U_i \\ 0 & \text{if } m \notin U_i \end{cases}$$

Show that λ_i is smooth.

Exercise 1.37 (!). (existence of partitions of unity)
Let $\{U_i,\ \varphi_i : U_i \to \mathbb{R}^n\}$ be a locally finite atlas on a manifold M such that $\varphi_i^{-1}(B_1)$ cover M as well (such an atlas was constructed in Exercise 1.33). Consider the functions λ_i's constructed in Exercise 1.36. Show that $\sum_j \lambda_j$ is well defined, vanishes nowhere, and that the family of functions $\left\{f_i := \frac{\lambda_i}{\sum_j \lambda_j}\right\}$ provides a partition of unity on M.

Remark 1.12. From this exercise it follows that any manifold with locally finite countable atlas admits a partition of unity.

Exercise 1.38 (*). Let M be a manifold admitting a countable atlas. Prove that M admits a countable, locally finite atlas, or find a counterexample.

Exercise 1.39 (**). Show that any Hausdorff, connected manifold admits a countable, locally finite atlas, or find a counterexample.

Exercise 1.40. Let M be a compact manifold, $\{V_i, \phi_i :\ V_i \longrightarrow \mathbb{R}^n, i = 1, 2, ..., m\}$ an atlas (which can be chosen finite), and $\nu_i :\ M \longrightarrow [0,1]$ the subordinate partition of unity.

a. (!) Consider the map $\Phi_i :\ M \longrightarrow \mathbb{R}^{n+1}$, with

$$\Phi_i(z) := \frac{(\nu_i\phi_i(z), 1)}{|\nu_i\phi_i(z)|^2 + 1}$$

Show that Φ_i is smooth, and its image lies in the n-dimensional sphere $S^n \subset \mathbb{R}^{n+1}$.

b. (*) Show that $\Phi_i :\ M \longrightarrow S^n$ is surjective.

c. (!) Let $U_i \subset V_i$ be the set where $\nu_i \neq 0$. Show that the restriction $\Phi_i\Big|_{V_i} :\ V_1 \longrightarrow S^n$ is an open embedding.

d. (!) Show that $\prod_{i=1}^m :\ \Phi_i :\ M \longrightarrow \underbrace{S^n \times S^n \times ... \times S^n}_{m \text{ times}}$ is a closed embedding.

Remark 1.13. We have just proved a weaker form of Whitney's theorem: each compact manifold admits a smooth embedding to $\mathbb{R}^N$.