Geometry 10: De Rham algebra

Rules: You may choose to solve only "hard" exercises (marked with !, * and **) or "ordinary" ones (marked with ! or unmarked), or both, if you want to have extra problems. To have a perfect score, a student must obtain (in average) a score of 10 points per week. It's up to you to ignore handouts entirely, because passing tests in class and having good scores at final exams could compensate (at least, partially) for the points obtained by grading handouts.

Solutions for the problems are to be explained to the examiners orally in the class and marked in the score sheet. It's better to have a written version of your solution with you. It's OK to share your solutions with other students, and use books, Google search and Wikipedia, we encourage it.

If you have got credit for 2/3 of ordinary problems or 2/3 of "hard" problems, you receive $6t$ points, where t is a number depending on the date when it is done. Passing all "hard" or all "ordinary" problems (except at most 2) brings you $10t$ points. Solving of "**" (extra hard) problems is not obligatory, but each such problem gives you a credit for 2 "*" or "!" problems in the "hard" set.

The first 3 weeks after giving a handout, $t = 1.5$, between 21 and 35 days, $t = 1$, and afterwards, $t = 0.7$. The scores are not cumulative, only the best score for each handout counts.

Please keep your score sheets until the final evaluation is given.

10.1 Kähler differentials

Definition 10.1. Let R be a ring over a field k, and V an R-module. A k-linear map $D: R \longrightarrow V$ is called **a derivation** if it satisfies **the Leibnitz identity** $D(ab) = aD(b) + bD(a)$. The space of derivations from R to V is denoted $\operatorname{Der}_k(R, V)$.

Exercise 10.1. Consider an action of R on $\operatorname{Der}_k(R, V)$, with rd acting as $a \longrightarrow rd(a)$. Prove that this defines a structure of R-module on $\operatorname{Der}_k(R, V)$.

Exercise 10.2. Let $[K : k]$ be a finite extension of a field of characteristic 0, and V a vector space over K. Prove that $\operatorname{Der}_k(K, V) = 0$.

Exercise 10.3 (!). Let M be a smooth manifold, $x \in M$ a point, $R = C^\infty M$, and $\mathfrak{m}_x \subset R$ the maximal ideal of x. Consider an R-module $V := R/\mathfrak{m}_x$. Find $\dim_{\mathbb{R}} \operatorname{Der}_R(R, V)$.

Exercise 10.4 (**). Let $R = C^0 M$ be a ring of continuous functions on a manifold M, and V an R-module of dimension 1 over $\mathbb{R}$. Find a non-trivial derivation $\nu \in \operatorname{Der}_k(R, V)$, or prove that it does not exist.

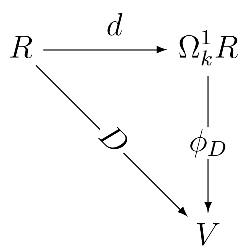
Definition 10.2. Let R be a ring over a field k. Define an R-module $\Omega^1_k R$, sometimes denoted simply as $\Omega^1 R$, with the following generators and relations. The generators of $\Omega^1_k R$ are indexed by elements of R; for each $a \in R$, the corresponding generator of $\Omega^1_k R$ is denoted da. Relations in $\Omega^1_k R$ are generated by expressions $d(ab) = adb + bda$, for all $a, b \in R$, and $d\lambda = 0$ for each $\lambda \in k$. Then $\Omega^1_k R$ is called **the module of Kähler differentials** of R.

Exercise 10.5. Prove that the natural map $R \longrightarrow \Omega^1_k R$, with $a \mapsto da$ is a derivation.

Exercise 10.6. Let R be a quotient of $k[r_1, ..., r_k]$ by an ideal. Prove that $\Omega^1_k R$ is generated as an R-module by $dr_1, ..., dr_k$.

Exercise 10.7 (!). Prove that $\operatorname{Der}_k(R) = \operatorname{Hom}_R(\Omega^1_k R, R)$.

Exercise 10.8. Let V be an R-module, and $D \in \operatorname{Der}_k(R, V)$ a derivation. Prove that there exists a unique R-module homomorphism $\phi_D : \Omega^1_k R \longrightarrow V$ making the following diagram commutative.



Remark 10.1. This property is often taken as a definition of $\Omega^1_k R$.

Exercise 10.9 (!). Let $R = k[t_1, ..., t_n]$ be a polynomial ring over a field of characteristic 0. Prove that $\Omega^1_k R$ is a free R-module generated by $dt_1, dt_2, ..., dt_n$.

Exercise 10.10 (*). Let $I \subset R$ be an ideal. Construct an exact sequence

$$I/I^2 \longrightarrow \Omega^1(R) \otimes_R R/I \longrightarrow \Omega^1(R/I) \longrightarrow 0.$$

Exercise 10.11. Let $R \xrightarrow{\phi} R'$ be a ring homomorphism. Consider $\Omega^1 R'$ as an R-module, using the action $r, a \longrightarrow \phi(r)a$.

a. Prove that there exists an R-module homomorphism $\Omega^1 R \longrightarrow \Omega^1 R'$, mapping dr to $d\phi(r)$.

b. Prove that it is unique.

Definition 10.3. In this case, we say that the homomorphism $\Omega^1 R \longrightarrow \Omega^1 R'$ **is induced by** ϕ.

Exercise 10.12 (*). Let R be a ring of continuous functions on a manifold, and $\mathfrak{m}_x$ a maximal ideal of a point. Prove that $\mathfrak{m}_x \Omega^1 R = \Omega^1 R$.

10.2 Cotangent bundle

Definition 10.4. Let A, B be R-modules, and $\nu: A \times B \longrightarrow R$ a bilinear pairing. It is called **non-degenerate** if for each $a \in A$ there exists $b \in B$ such that $\nu(a,b) \neq 0$, and for each $b \in B$ there exists $a \in A$ such that $\nu(a,b) \neq 0$

Exercise 10.13. Let A, B be vector spaces over k, and $\nu: A \times B \longrightarrow k$ a non-degenerate pairing. Prove that A is isomorphic to B^*, or find a counterexample

a. When A, B are finite-dimensional.

b. When A, B are infinite-dimensional.

Definition 10.5. Let V be an R-module. A **dual R-module** $\operatorname{Hom}_R(V, R)$ is denoted R^*.

Exercise 10.14. Let V be an R-module. Consider the natural pairing $V \times V^* \longrightarrow R$. Prove that it is non-degenerate, or find a counterexample, in the following cases:

a. when R is a field, and V a (possibly infinite-dimensional) vector space

b. (!) when the natural map $V \longrightarrow V^{**}$ is injective

c. when V is a free R-module

d. (*) when R is a ring which has no zero divisors.

Exercise 10.15 (*). Let A, B be finitely-generated R-modules, and $\nu: A \times B \longrightarrow R$ a non-degenerate pairing. Prove that A is isomorphic to B^*, or find a counterexample.

Exercise 10.16 (!). Let A be a free, finitely generated R-module, and $\nu: A \times B \longrightarrow R$ a non-degenerate pairing. Prove that B is also free, and isomorphic to A^*.

Definition 10.6. Let A, B be finitely generated R-modules, and $\nu: A \times B \longrightarrow R$ a bilinear pairing. Define **the annihilator of ν in** B as a submodule consisting of all elements $b \in B$ for which the homomorphism $\nu(\cdot, b): A \longrightarrow R$ vanishes.

Definition 10.7. Let M be a smooth manifold, $R := C^\infty M$ the ring of smooth finctions, and $\nu : \operatorname{Der}(R) \times \Omega^1 R \longrightarrow R$ the pairing constructed in Exercise 10.7. Consider its annihilator $K \subset \Omega^1 R$. Define **the cotangent bundle** as $\Lambda^1 M := \Omega^1 R/K$. For the purpose of this definition, $\Lambda^1 M$ is a $C^\infty M$-module.

Exercise 10.17. Let $R := C^\infty \mathbb{R}^n$, and $t_1, ..., t_n \in R$ be coordinate functions. Consider an element in $\Lambda^1 \mathbb{R}^n$, written as $P = \sum_{i=1}^n P_i dt_i$, let $Q = \sum_{i=1}^n Q_i \frac{d}{dt_i} \in \operatorname{Der}_k(R)$ – be a vector field, and $\nu : \operatorname{Der}(R) \times \Lambda^1 \mathbb{R}^n \longrightarrow R$ the natural pairing. Prove that $\nu(P, Q) = \sum_i P_i Q_i$.

Exercise 10.18 (!). In these assumptions, prove that $\Lambda^1 R$ is a free $\mathbb{R}$-module, generated by $dt_1, ..., dt_n$.

Hint. Prove that $\operatorname{Der}(R) = \operatorname{Hom}_R(\Omega^1 R, R)$, and $\operatorname{Der}(R)$ is a free R-module. Use exercise 10.16.

Exercise 10.19. Let A, B be finitely-generated projective R-modules, and $\nu : A \times B \longrightarrow R$ a non-degenerate pairing. Prove that $B \cong A^*$.

Exercise 10.20 (!). Let M be a smooth, metrizable manifold. Prove that

$$\Lambda^1 M = \operatorname{Hom}_{C^\infty M}(\operatorname{Der}(C^\infty M), C^\infty M).$$

Hint. Use the previous exercise and apply the Serre-Swan theorem.

Exercise 10.21 (*). Let K be the kernel of the natural projection

$$\Omega^1(C^\infty M) \longrightarrow \Lambda^1 M.$$

Prove that $\mathfrak{m}_x K = K$ for each maximal ideal of a point $x \in M$.

Exercise 10.22 (**). Show that K is non-empty.

10.3 De Rham algebra

Definition 10.8. Let M be a smooth manifold. **A bundle of differential i-forms on M** is the bundle $\Lambda^i T^* M$ of antisymmetric i-forms on TM. It is denoted $\Lambda^i M$.

Definition 10.9. Let $\alpha \in (V^*)^{\otimes i}$ and $\alpha \in (V^*)^{\otimes j}$ be polylinear forms on V. Define the **tensor multiplication** $\alpha \otimes \beta$ as

$$\alpha \otimes \beta(x_1, ..., x_{i+j}) := \alpha(x_1, ..., x_j)\beta(x_{i+1}, ..., x_{i+j}).$$

Exercise 10.23. Let $\bigotimes_k T^*M \xrightarrow{\Pi} \Lambda^k M$ be the antisymmetrization map,

$$\Pi(\alpha)(x_1, ..., x_n) := \frac{1}{n!} \sum_{\sigma \in \mathsf{Sym}_n} (-1)^\sigma \alpha(x_{\sigma_1}, x_{\sigma_2}, ..., x_{\sigma_n}).$$

Define the "exterior multiplication" $\wedge : \Lambda^i M \times \Lambda^j M \longrightarrow \Lambda^{i+j} M$ as $\alpha \wedge \beta := \Pi(\alpha \otimes \beta)$, where $\alpha \otimes \beta$ is a section $\Lambda^i M \otimes \Lambda^j M \subset \bigotimes_{i+j} T^*M$ obtained as their tensor multiplication. Prove that this operation is associative and satisfies $\alpha \wedge \beta = (-1)^{ij} \beta \wedge \alpha$.

Definition 10.10. The algebra $\Lambda^* M := \oplus_i \Lambda^i M$ with the multiplicative structure defined above is called **the de Rham algebra** of a manifold.

Exercise 10.24 (*). Let M be an oriented manifold. Prove that all bundles $\Lambda^i M$ are oriented, or find a counterexample.

Exercise 10.25. Prove that de Rham algebra is multiplicatively generated by $C^\infty M = \Lambda^0 M$ and $d(C^\infty) \subset \Lambda^1 M$.

Exercise 10.26. Prove that a derivation on an algebra is uniquely determined by its values on any set of multiplicative generators of this algebra.

Definition 10.11. De Rham differential $d : \Lambda^* M \longrightarrow \Lambda^{*+1} M$ is an $\mathbb{R}$-linear map satisfying the following conditions.

(i) For each $f \in \Lambda^0 M = C^\infty M$, $d(f) \in \Lambda^1 M$ is equal to the image of the Kähler differential $df \in \Omega^1 M$ in $\Lambda^1 M = \Omega^1 M/K$.

(ii) (Leibnitz rule) $d(a \wedge b) = da \wedge b + (-1)^j a \wedge db$ for any $a \in \Lambda^i M, b \in \Lambda^j M$.

(iii) $d^2 = 0$.

Exercise 10.27 (!). Prove that de Rham differential is defined uniquely by these axioms.

Hint. Use the previous exercise.

Exercise 10.28. Let $t_1, ..., t_n$ be coordinate functions on $\mathbb{R}^n$, and $\alpha \in \Lambda^*\mathbb{R}^n$ a monomial obtained as a product of several dt_i,

$$\alpha = dt_{i_1} \wedge dt_{i_2} \wedge ... \wedge dt_{i_k},$$

$i_1 < i_2 < ... < i_k$ (such a monomial is called **a coordinate monomial**).

a. Prove that $\Lambda^*\mathbb{R}^n$ is a trivial bundle, and coordinate monomials are free generators of $\Lambda^*\mathbb{R}^n$.

b. Show that the de Rham differential, if it exists, satisfies $d(f\alpha) = \sum_i \frac{df}{dt_i} dt_i \wedge \alpha$ for any $f \in C^\infty\mathbb{R}^n$.

c. Prove that this formula defines the de Rham differential on $\Lambda^*\mathbb{R}^n$ correctly.

Exercise 10.29. a. Prove that de Rham differential $d: \Lambda^*M \longrightarrow \Lambda^{*+1}M$ commutes with restrictions to open subsets.

b. Show that de Rham differential (if it exists) defines a sheaf morphism.

Hint. Use uniqueness of de Rham differential.

Exercise 10.30 (!). Prove that de Rham differential exists on any manifold.

Hint. Locally, de Rham differential is constructed in exercise 10.28. To go from local to global, use the previous exercise, and apply the sheaf axioms.

Exercise 10.31 (*). Let R be a ring over a field, and $\Omega^i R := \Lambda^i_R \Omega^1 R$ an exterior algebra generated by Kähler differentials. Prove that there exists the de Rham differential $d: \Omega^* R \longrightarrow \Omega^{*+1} R$ satisfying the axioms above.