Geometry 11: Lie derivatives and Poincaré lemma

Rules: You may choose to solve only "hard" exercises (marked with !, * and **) or "ordinary" ones (marked with ! or unmarked), or both, if you want to have extra problems. To have a perfect score, a student must obtain (in average) a score of 10 points per week. It's up to you to ignore handouts entirely, because passing tests in class and having good scores at final exams could compensate (at least, partially) for the points obtained by grading handouts.

Solutions for the problems are to be explained to the examiners orally in the class and marked in the score sheet. It's better to have a written version of your solution with you. It's OK to share your solutions with other students, and use books, Google search and Wikipedia, we encourage it.

If you have got credit for 2/3 of ordinary problems or 2/3 of "hard" problems, you receive $6t$ points, where t is a number depending on the date when it is done. Passing all "hard" or all "ordinary" problems (except at most 2) brings you $10t$ points. Solving of "**" (extra hard) problems is not obligatory, but each such problem gives you a credit for 2 "*" or "!" problems in the "hard" set.

The first 3 weeks after giving a handout, $t = 1.5$, between 21 and 35 days, $t = 1$, and afterwards, $t = 0.7$. The scores are not cumulative, only the best score for each handout counts.

Please keep your score sheets until the final evaluation is given.

11.1 Lie derivative

Definition 11.1. An associative algebra $A^* = \oplus_{i\in\mathbb{Z}} A^i$ is called **a graded algebra** if for all $a \in A^i$, $b \in A^j$, the product ab lies in A^{i+j}.

Definition 11.2. Let $A^* = \oplus_{i\in\mathbb{Z}} A^i$ be a graded algebra over a field. It is called **graded commutative**, or **supercommutative**, if $ab = (-1)^{ij} ba$ for all $a \in A^i, b \in A^j$.

Remark 11.1. Grassmann algebra $\Lambda^* V$ is clearly supercommutative.

Exercise 11.1. Let A^*, B^* be graded commutative algebras, and $A^* \otimes B^*$ their tensor product, with a grading $(A^* \otimes B^*)^p := \oplus_{i+j=p} A^i \otimes B^j$, and multiplication, defined as $a \otimes b \cdot a' \otimes b' = (-1)^{ij} aa' \otimes bb'$, where $a' \in A^i, b \in B^j$. Prove that it is supercommutative.

Exercise 11.2. Let V, W be vector spaces, and $A^* := \Lambda^* V, B^* := \Lambda^* W$ their Grassmann algebras. Prove that $\Lambda^*(V \oplus W)$ is isomorphic to a tensor product $A^* \otimes B^*$, defined as above.

Definition 11.3. Let A^* be a graded commutative algebra, and $D: A^* \longrightarrow A^{*+i}$ be a map which shifts grading by i. It is called a **graded derivation** if $D(ab) = D(a)b + (-1)^{ij} aD(b)$, for each $a \in A^j$.

Remark 11.2. If i is even, graded derivation is just a derivation. If it is even, it is called **odd derivation**.

Remark 11.3. De Rham differential is an odd derivation, by definition.

Definition 11.4. Let M be a smooth manifold, and $X \in TM$ a vector field. Consider an operation of **convolution with a vector field** $i_X: \Lambda^i M \longrightarrow \Lambda^{i-1} M$, mapping an i-form α to an $(i-1)$-form $v_1, ..., v_{i-1} \longrightarrow \alpha(X, v_1, ..., v_{i-1})$

Exercise 11.3. Prove that i_X is an odd derivation.

Exercise 11.4 (*). Let $D: A^* \longrightarrow A^{*+i}$ be a linear map such that for all $x \in A$ there exists N such that $D^N(x) = 0$. Prove that $e^D := 1 + D + \frac{D^2}{2} + ... + \frac{D^i}{i!} + ...$ is an automorphism of A^* if and only if D is a derivation.

Definition 11.5. Let A^* be a graded vector space, and $E: A^* \longrightarrow A^{*+i}$, $F: A^* \longrightarrow A^{*+j}$ operators shifting the grading by i, j. Define **the supercommutator** by the formula

$$\{E, F\} := EF - (-1)^{ij} FE.$$

Remark 11.4. An endomorphism which shifts a grading by i is called **even** if i is even, and **odd** otherwise.

Exercise 11.5. Prove that a supercommutator satisfies **graded Jacobi identity,**

$$\{E, \{F, G\}\} = \{\{E, F\}, G\} + (-1)^{\tilde{E}\tilde{F}} \{F, \{E, G\}\}$$

where $\tilde{E}$ and $\tilde{F}$ are 0 if E, F are even, and 1 otherwise.

Remark 11.5. There is a simple mnemonic rule which allows one to remember a superidentity, if you know the commutative analogue. Each time when in commutative case two letters A, F are exchanged, in supercommutative case one needs to multiply by $(-1)^{\tilde{E}\tilde{F}}$.

Exercise 11.6. Let A^* be a graded commutative algebra and $a \in A$. Denote by $L_a : A \longrightarrow A$ the operation of multiplication by a: $L_a(b) = ab$. Prove that D is a superderivation if and only if $D(1) = 0$ and for each $a \in A^i$, the supercommutator $\{D, L_a\}$ is equal to L_b for some $b \in A^*$.

Exercise 11.7 (!). Prove that a supercommutator of superderivations is again a superderivation.

Hint. Use the Jacobi identity and apply the previous exercise.

Definition 11.6. Let B be a smooth manifold, and $v \in TM$ a vector field. An endomorphism $\operatorname{Lie}_v : \Lambda^* M \longrightarrow \Lambda^* M$, preserving the grading is called **a Lie derivative along** v if it satisfies the following conditions.

(i) On functions Lie_v is equal to a derivative along v.

(ii) $[\operatorname{Lie}_v, d] = 0$

(iii) Lie_v is a derivation on the de Rham algebra.

Exercise 11.8. Let $\nu_1, \nu_2 : \Lambda^*(M) \longrightarrow \Lambda^*(M)$ be derivations of the de Rham algebra. Suppose that ν_1 is equal to ν_2 on $C^\infty M = \Lambda^0(M)$ and on $d(C^\infty M)$. Prove that $\nu_1 = \nu_2$.

Hint. $\Lambda^*(M)$ is generated (multiplicatively) by $C^\infty M = \Lambda^0(M)$ and $d(C^\infty M)$.

Exercise 11.9. Prove that the Lie derivative is uniquely determined by the properties (i)-(iii).

Hint. Use the previous exercise.

Exercise 11.10. Prove that $\{d, \{d, E\}\} = 0$, for each $E \in \operatorname{End}(\Lambda^* M)$.

Hint. Use the graded Jacobi identity.

Exercise 11.11. Prove that $\{d, i_v\}$ commutes with d, where $i_v: \ \Lambda^* M \longrightarrow \Lambda^{*-1} M$ is a convolution with v.

Hint. Use the previous exercise.

Exercise 11.12 (!). (Cartan formula) Prove that $\{d, i_v\}$ is a Lie derivative along v.

Exercise 11.13 (*). Let $\tau: \ \Lambda^*(M) \longrightarrow \Lambda^{*-1}(M)$ be a derivation shifting grading by -1. Prove that there exists a vector field $v \in TM$ such that $\tau = i_v$, or find a counterexample.

11.2 Poincaré lemma

Exercise 11.14. Let t be the coordinate function on a real line, $f(t) \in C^\infty \mathbb{R}$ a smooth function, and $v := t\frac{d}{dt}$ a vector field. Define

$$R(f)(t) := \int_0^1 \frac{f(\lambda t)}{\lambda} d\lambda$$

Prove that this integral converges whenever $f(0) = 0$, and satisfies $\operatorname{Lie}_v R(f) = f$ in this case.

Exercise 11.15. Let $t_1, ..., t_n$ be coordinate functions in $\mathbb{R}^n$, and $\vec{r} := \sum_i t_i \frac{d}{dt_i}$ a radial vector field. Consider a function $f \in C^\infty \mathbb{R}^n$ satisfying $f(0) = 0$, and let $x = (x_1, ..., x_n)$ be any point in $\mathbb{R}^n$. Prove that an integral

$$R(f)(x) := \int_0^1 \frac{f(\lambda x)}{\lambda} d\lambda$$

converges, and satisfies $\operatorname{Lie}_{\vec{r}} R(f) = f$.

Hint. Use the previous exercise.

Definition 11.7. An open subset $U \subset \mathbb{R}^n$ is called **starlike** if for any $x \in U$ the interval $[0, x]$ belongs to U.

Exercise 11.16 (!). Let U be a starlike subset in $\mathbb{R}^n$, and $i > 0$. Construct an operator

$$R: \ \Lambda^i U \longrightarrow \Lambda^i U$$

which satisfies $\operatorname{Lie}_{\vec{r}} R\alpha = R \operatorname{Lie}_{\vec{r}} \alpha = \alpha$ for each $\alpha \in \Lambda^i U$.

Hint. Define the integral $R(\alpha)$ as in the previous exercise, and check that it converges. Prove that $\operatorname{Lie}_{\vec{r}} R(\alpha) = \alpha$.

Exercise 11.17. Prive that any form $\alpha \in \Lambda^i U$ on a starlike set U satisfying $\operatorname{Lie}_{\vec{r}} \alpha = 0$ vanishes if $i > 0$.

Hint. Use the previous exercise.

Exercise 11.18 (!). Prove that $\{R, d\} = 0$

Hint. Check that

$$\{R, d\} \operatorname{Lie}_{\vec{r}} \alpha = Rd \operatorname{Lie}_{\vec{r}} \alpha + dR \operatorname{Lie}_{\vec{r}} \alpha = -R \operatorname{Lie}_{\vec{r}} d\alpha + d\alpha = 0.$$

For any $\beta \in \mathsf{ker}\{R, d\} *^{\rangle} \mathcal{M}$, satisfying $i > 0$ or $\beta(0) = 0$ for $i = 0$, solve an equation $\operatorname{Lie}_{\vec{r}} \alpha = \beta$.

Exercise 11.19. Prove that $\{d, i_{\vec{r}}\} R(\alpha) = \alpha$, for any i-form α on a starlike set, $i > 0$.

Definition 11.8. Let d be de Rham differential. A form in $\mathsf{ker}\, d$ is called **closed**, a form in $\mathsf{im}\, d$ is called **exact**. Since $d^2 = 0$, any exact form is closed. **The group of i-th de Rham cohomology of** M, denoted $H^i(M)$, is a quotient of a space of closed i-forms by exact: $H^*(M) = \frac{\mathsf{ker}\, d}{\mathsf{im}\, d}$.

Exercise 11.20 (!). Let $\alpha \in \Lambda^i U$ be a closed i-form on a starlike set U, with $i > 0$. Prove that $\alpha = d i_{\vec{r}} R(\alpha)$.

Hint. Use the previous exercise.

Exercise 11.21 (!). (Poincaré lemma) Let U be a starlike set. Prove that $H^i(U) = 0$ for each $i > 0$, and $H^0(M) = \mathbb{R}$.

Exercise 11.22. Let θ be a closed form, and $d_\theta(x) = dx + \theta \wedge x$ the corresponding operator on $\Lambda^* M$. Its **cohomology** are defined as $H^*(\Lambda^*(M), d_\theta) := \frac{\mathsf{ker}\, d_\theta}{\mathsf{im}\, d_\theta}$

a. Show that $d_\theta^2 = 0$.

b. (*) Let θ be an exact 1-form. Prove that $H^i(\Lambda^*(M), d_\theta)$ are isomorphic to $H^i(M)$.

c. (*) Let θ be a closed 1-form. Prove that $H^i(\Lambda^*(M), d_\theta)$ are isomorphic to $H^i(M)$, or find a counterexample.

d. (**) Let θ be a closed 3-form. Prove that $H^i(\Lambda^*(M), d_\theta)$ are isomorphic to $H^i(M)$, or find a counterexample.

11.3 Pullback of a differential form

Definition 11.9. Let $M \xrightarrow{\phi} N$ be a morphism of smooth manifolds, and $\Lambda^1 N \xrightarrow{\phi^*} \Lambda^1 M$ an induced morphism which maps fdg to $\phi^* f d\phi^* g$. A form $\phi^*\alpha$ is called **pullback of** α.

Exercise 11.23. Prove that ϕ^* can be extended from $\Lambda^1 N$ to a multiplicative homomorphism

$$\phi^* : \Lambda^* N \longrightarrow \Lambda^* M.$$

Definition 11.10. Pullback of an i-form α is a form $\phi^*\alpha$ defined as above. If $M \xrightarrow{\phi} N$ is a closed embedding, the form $\phi^*\alpha$ is called **restriction** of α to $M \hookrightarrow N$.

Exercise 11.24. Let $x \in T_m M$ be a tangent vector, and $\alpha \in \Lambda^1 N$ a 1-form. Prove that $\phi^*\alpha(x) = \alpha\left(D_\phi(x)\right)$, where $D_\phi : T_m M \longrightarrow T_{\phi(m)} N$ is a differential.

Exercise 11.25 (!). Prove that $\phi^* d\alpha = d\phi^*\alpha$.

Hint. Use exercise 11.8.

Definition 11.11. Let $f : M \times [a, b] \longrightarrow M$ be a smooth map such that for all $t \in [a, b]$ the restriction $f_t := f\Big|_{M\times\{t\}} : M \longrightarrow M$ is a diffeomorphism. Then f is called **a flow of diffeomorphisms**.

Exercise 11.26. Let V_t be a flow of diffeomorphisms, $f \in C^\infty M$, and $V_t^*(f)(x) := f(V_t(x))$. Consider the map $\frac{d}{dt}V_t\Big|_{t=c} : C^\infty M \longrightarrow C^\infty M$, with $\frac{d}{dt}V_t\Big|_{t=c}(f) = (V_c^{-1})^* \frac{dV_t}{dt}\Big|_{t=c} f$. Show that $\frac{d}{dt}V_t\Big|_{t=c}$ is a derivation (that is, a vector field).

Definition 11.12. The vector field $\frac{d}{dt}V_t\Big|_{t=c}$ is called **a vector field tangent to a flow of diffeomorphisms V_t at $t = c$.**

Definition 11.13. Let v be a vector field on M, and $V : M \times [a, b] \longrightarrow M$ a flow of diffeomorphisms which satisfies $\frac{d}{dt}V_t\Big|_{t=c} = v$ for each c, and $V_0 = \mathsf{Id}$. Then V_t is called **an exponent of** v.

Exercise 11.27. Prove that an exponent of a vector field is unique, if it exists.

Exercise 11.28 (*). Let v be a vector field, and $D_v : C^\infty M \longrightarrow C^\infty M$ the corresponding derivation. Suppose that the serie $e^{tD_v}(f) := \sum_{i=0}^\infty \frac{t^i}{i!}(D_v)^i f$ converges for some $f \in C^\infty M$ and $t \in [0, a]$, and the exponent V_t of v exists for all $t \in [0, a]$. Prove that $e^{tD_v}(f) = V_t^*(f)$.

Exercise 11.29 (!). Let v be a vector field, and V_t its exponent. For any $\alpha \in \Lambda^* M$, consider $V_t^*\alpha$ as a $\Lambda^* M$-valued function of t. Prove that $\operatorname{Lie}_v \alpha = \frac{d}{dt}(V_t^*\alpha)$.

Hint. Use exercise 11.8.

11.4 Computation of cohomology

Exercise 11.30 (*). Let M be a smooth manifold equipped with a smooth action X_t of the group $S^1 = U(1)$, $f \in C^\infty M$, and $f_t := X_t^* f$. Prove that $f_t = \sum_{n\in\mathbb{Z}} e^{2\pi\sqrt{-1}nt} f_i$, where $f_i \in C^\infty M$ are S^1-invariant functions.

Definition 11.14. Let $A \subset \Lambda^* M$ be a subspace which satisfies $d(A) \subset A$. **Cohomology** of A is the quotient $H^*(A) := \frac{\ker d\big|_A}{d(A)}$. If A is graded, $A = \bigoplus A^i$, where $A_i := \Lambda^i(M) \cap A$, one has $H^*(A) = \bigoplus_i H^i(A)$, where $H^i(A) = \frac{\ker d\big|_A{}^i}{d(A^{i-1})}$.

Exercise 11.31. Let M be a smooth manifold, and $v \in TM$ a vector field. Prove that the image $\operatorname{Lie}_v \Lambda^* M$ satisfies $d(\operatorname{Lie}_v \Lambda^* M) \subset \operatorname{Lie}_v \Lambda^* M$, and $H^*(\operatorname{Lie}_v \Lambda^* M) = 0$.

Exercise 11.32. Let M be a smooth manifold, and $v \in TM$ a vector field. Consider the space $\Lambda^*_v M := \ker \operatorname{Lie}_v \big|_{\Lambda^* M}$. Let V_t be an exponent of v. Suppose that V_t is periodic, $V_{t+2\pi} = V_t$.

a. (!) Prove that $(\operatorname{Lie}_v \Lambda^* M) \cap \Lambda^*_v M = \emptyset$.

b. (*) Prove that $(\operatorname{Lie}_v \Lambda^* M) \oplus \Lambda^*_v M = \Lambda^* M$.

Hint. Use exercise 11.30.

Exercise 11.33. Let M be a smooth manifold, and $v \in TM$ a vector field. Suppose that $(\operatorname{Lie}_v \Lambda^* M) \oplus \Lambda^*_v M = \Lambda^* M$. Prove that $H^*(M) = H^*(\Lambda^*_v M)$.

Exercise 11.34. Let M be a smooth manifold, $K \subset \mathbb{R}^n$ a starlike set, and v the radial vector field on K lifted to $M \times K$.

a. (!) Prove that $\Lambda^*_v(M \times K) = \Lambda^*(M)$.

b. Prove that $H^*(M \times K) = H^*(M)$.

Exercise 11.35. Let v_1, v_2 be vector fields which satisfy $\operatorname{Lie}_{v_i} \Lambda^* M \oplus \Lambda^* M_{v_i} = \Lambda^* M$.

a. Prove that
$$\Lambda^* M = (\ker \operatorname{Lie}_{v_1} \cap \ker \operatorname{Lie}_{v_2}) \oplus (\operatorname{im} \operatorname{Lie}_{v_1} \cup \operatorname{im} \operatorname{Lie}_{v_2}).$$

b. (!) Prove that $H^*(\operatorname{im} \operatorname{Lie}_{v_1} \cup \operatorname{im} \operatorname{Lie}_{v_2}) = 0$.

Exercise 11.36 (*). Let G be a Lie group (that is, a smooth manifold equipped with a smooth group structure) generated by circles $S^1 \subset G$, M a manifold with G-action, and $\Lambda^*_G M$ the space of G-invariant forms. Prove that $H^*(\Lambda^*_G M) = H^* M$.

Hint. Use the previous exercise.

Exercise 11.37 (*). Compute cohomology of a sphere S^n and a real projective space $\mathbb{R}P^n$.

Hint. Use the previous exercise.