### 11.1 Lie derivative

**Definition 11.1.** An associative algebra $A^* = \oplus_{i \in \mathbb{Z}} A^i$ is called a **graded algebra** if for all $a \in A^i$, $b \in A^j$, the product $ab$ lies in $A^{i+j}$.

**Definition 11.2.** Let $A^* = \oplus_{i \in \mathbb{Z}} A^i$ be a graded algebra over a field. It is called **graded commutative**, or **supercommutative**, if $ab = (-1)^{ij} ba$ for all $a \in A^i, b \in A^j$.

**Remark 11.1.** Grassmann algebra $\Lambda^* V$ is clearly supercommutative.

**Exercise 11.1.** Let $A^*, B^*$ be graded commutative algebras, and $A^* \otimes B^*$ their tensor product, with a grading $(A^* \otimes B^*)^p := \oplus_{i+j=p} A^i \otimes B^j$, and multiplication, defined as $a \otimes b \cdot a' \otimes b' = (-1)^{ij} a a' \otimes b b'$, where $a', b \in B^j$. Prove that it is supercommutative.

**Exercise 11.2.** Let $V, W$ be vector spaces, and $A^* := \Lambda^* V, B^* := \Lambda^* W$ their Grassmann algebras. Prove that $\Lambda^*(V \oplus W)$ is isomorphic to a tensor product $A^* \otimes B^*$, defined as above.

**Definition 11.3.** Let $A^*$ be a graded commutative algebra, and $D : A^* \to A^{*+i}$ be a map which shifts grading by $i$. It is called a **graded derivation** if $D(ab) = D(a)b + (-1)^{ij} aD(b)$, for each $a \in A^i$.

**Remark 11.2.** If $i$ is even, graded derivation is just a derivation. If it is even, it is called **odd derivation**.

**Remark 11.3.** De Rham differential is an odd derivation, by definition.

**Definition 11.4.** Let $M$ be a smooth manifold, and $X \in TM$ a vector field. Consider an operation of **convolution with a vector field** $i_X : \Lambda^i M \to \Lambda^{i-1} M$, mapping an $i$-form $\alpha$ to an $(i-1)$-form $v_1, ..., v_{i-1} \mapsto \alpha(X, v_1, ..., v_{i-1})$

**Exercise 11.3.** Prove that $i_X$ is an odd derivation.
Exercise 11.4 (*). Let $D : A^* \longrightarrow A^{*+i}$ be a linear map such that for all $x \in A$ there exists $N$ such that $D^N(x) = 0$. Prove that $e^D := 1 + D + \frac{D^2}{2} + \ldots + \frac{D^i}{i!} + \ldots$ is an automorphism of $A^*$ if and only if $D$ is a derivation.

Definition 11.5. Let $A^*$ be a graded vector space, and $E : A^* \longrightarrow A^{*+i}, F : A^* \longrightarrow A^{*+j}$ operators shifting the grading by $i,j$. Define the supercommutator by the formula

$$\{E, F\} := EF - (-1)^{ij} FE.$$

Remark 11.4. An endomorphism which shifts a grading by $i$ is called even if $i$ is even, and odd otherwise.

Exercise 11.5. Prove that a supercommutator satisfies graded Jacobi identity,

$$\{E, \{F, G\}\} = \{\{E, F\}, G\} + (-1)^{\tilde{E}\tilde{F}} \{F, \{E, G\}\}$$

where $\tilde{E}$ and $\tilde{F}$ are 0 if $E, F$ are even, and 1 otherwise.

Remark 11.5. There is a simple mnemonic rule which allows one to remember a superidentity, if you know the commutative analogue. Each time when in commutative case two letters $A$, $F$ are exchanged, in supercommutative case one needs to multiply by $(-1)^{\tilde{E}\tilde{F}}$.

Exercise 11.6. Let $A^*$ be a graded commutative algebra and $a \in A$. Denote by $L_a : A \longrightarrow A$ the operation of multiplication by $a$: $L_a(b) = ab$. Prove that $D$ is a superderivation if and only if $D(1) = 0$ and for each $a \in A^1$, the supercommutator $\{D, L_a\}$ is equal to $L_b$ for some $b \in A^*$.

Exercise 11.7 (!). Prove that a supercommutator of superderivations is again a superderivation.

Hint. Use the Jacobi identity and apply the previous exercise.

Definition 11.6. Let $B$ be a smooth manifold, and $v \in TM$ a vector field. An endomorphism $\text{Lie}_v : \Lambda^* M \longrightarrow \Lambda^* M$, preserving the grading is called a Lie derivative along $v$ if it satisfies the following conditions.

(i) On functions $\text{Lie}_v$ is equal to a derivative along $v$.

(ii) $[\text{Lie}_v, d] = 0$

(iii) $\text{Lie}_v$ is a derivation on the de Rham algebra.

Exercise 11.8. Let $\nu_1, \nu_2 : \Lambda^*(M) \longrightarrow \Lambda^*(M)$ be derivations of the de Rham algebra. Suppose that $\nu_1$ is equal to $\nu_2$ on $C^\infty M = \Lambda^0(M)$ and on $d(C^\infty M)$. Prove that $\nu_1 = \nu_2$.

Hint. $\Lambda^*(M)$ is generated (multiplicatively) by $C^\infty M = \Lambda^0(M)$ and $d(C^\infty M)$. 

Exercise 11.9. Prove that the Lie derivative is uniquely determined by the properties (i)-(iii).

Hint. Use the previous exercise.

Exercise 11.10. Prove that \( \{d, \{d, E\}\} = 0 \), for each \( E \in \text{End}(\Lambda^* M) \).

Hint. Use the graded Jacobi identity.

Exercise 11.11. Prove that \( \{d, i_v\} \) commutes with \( d \), where \( i_v : \Lambda^* M \to \Lambda^{*-1} M \) is a convolution with \( v \).

Hint. Use the previous exercise.

Exercise 11.12 (!). (Cartan formula) Prove that \( \{d, i_v\} \) is a Lie derivative along \( v \).

Exercise 11.13 (*). Let \( \tau : \Lambda^* (M) \to \Lambda^{*-1} (M) \) be a derivation shifting grading by \(-1\). Prove that there exists a vector field \( v \in TM \) such that \( \tau = i_v \), or find a counterexample.

11.2 Poincaré lemma

Exercise 11.14. Let \( t \) be the coordinate function on a real line, \( f(t) \in C^\infty \mathbb{R} \) a smooth function, and \( v := t \frac{d}{dt} \) a vector field. Define

\[
R(f)(t) := \int_0^1 \frac{f(\lambda t)}{\lambda} d\lambda
\]

Prove that this integral converges whenever \( f(0) = 0 \), and satisfies \( \text{Lie}_v R(f) = f \) in this case.

Exercise 11.15. Let \( t_1, ..., t_n \) be coordinate functions in \( \mathbb{R}^n \), and \( \vec{r} := \sum \frac{t_i}{t_i} \frac{d}{dt_i} \) a radial vector field. Consider a function \( f \in C^\infty \mathbb{R}^n \) satisfying \( f(0) = 0 \), and let \( x = (x_1, ..., x_n) \) be any point in \( \mathbb{R}^n \). Prove that an integral

\[
R(f)(x) := \int_0^1 \frac{f(\lambda x)}{\lambda} d\lambda
\]

converges, and satisfies \( \text{Lie}_{\vec{r}} R(f) = f \).

Hint. Use the previous exercise.

Definition 11.7. An open subset \( U \subset \mathbb{R}^n \) is called starlike if for any \( x \in U \) the interval \([0, x]\) belongs to \( U \).

Exercise 11.16 (!). Let \( U \) be a starlike subset in \( \mathbb{R}^n \), and \( i > 0 \). Construct an operator \( R : \Lambda^i U \to \Lambda^i U \) which satisfies \( \text{Lie}_{\vec{r}} R \alpha = R \text{Lie}_{\vec{r}} \alpha = \alpha \) for each \( \alpha \in \Lambda^i U \).
Hint. Define the integral $R(\alpha)$ as in the previous exercise, and check that it converges. Prove that $\text{Lie}_\vec{r} R(\alpha) = \alpha$.

**Exercise 11.17.** Prove that any form $\alpha \in \Lambda^i U$ on a starlike set $U$ satisfying $\text{Lie}_\vec{r} \alpha = 0$ vanishes if $i > 0$.

**Hint.** Use the previous exercise.

**Exercise 11.18 (!).** Prove that $\{R, d\} = 0$

**Hint.** Check that $\{R, d\} \text{Lie}_\vec{r} \alpha = R d \text{Lie}_\vec{r} \alpha + d R \text{Lie}_\vec{r} \alpha = -R \text{Lie}_\vec{r} d \alpha + d\alpha = 0$.

For any $\beta \in \ker \{R, d\} \ast M$, satisfying $i > 0$ or $\beta(0) = 0$ for $i = 0$, solve an equation $\text{Lie}_\vec{r} \alpha = \beta$.

**Exercise 11.19.** Prove that $\{d, i_\vec{r}\} R(\alpha) = \alpha$, for any $i$-form $\alpha$ on a starlike set, $i > 0$.

**Definition 11.8.** Let $d$ be de Rham differential. A form in $\ker d$ is called closed, a form in $\text{im} d$ is called exact. Since $d^2 = 0$, any exact form is closed. **The group of $i$-th de Rham cohomology** of $M$, denoted $H^i(M)$, is a quotient of a space of closed $i$-forms by exact: $H^i(M) = \ker d \text{im} d$.

**Exercise 11.20 (!).** Let $\alpha \in \Lambda^i U$ be a closed $i$-form on a starlike set $U$, with $i > 0$. Prove that $\alpha = d i_\vec{r} R(\alpha)$.

**Hint.** Use the previous exercise.

**Exercise 11.21 (!).** (Poincaré lemma) Let $U$ be a starlike set. Prove that $H^i(U) = 0$ for each $i > 0$, and $H^0(M) = \mathbb{R}$.

**Exercise 11.22.** Let $\theta$ be a closed form, and $d_\theta(x) = dx + \theta \wedge x$ the corresponding operator on $\Lambda^* M$. Its cohomology are defined as $H^i(\Lambda^*(M), d_\theta) := \ker d_\theta \text{im} d_\theta$.

a. Show that $d_\theta^2 = 0$.

b. (*) Let $\theta$ be an exact 1-form. Prove that $H^i(\Lambda^*(M), d_\theta)$ are isomorphic to $H^i(M)$.

c. (*) Let $\theta$ be a closed 1-form. Prove that $H^i(\Lambda^*(M), d_\theta)$ are isomorphic to $H^i(M)$, or find a counterexample.

d. (**) Let $\theta$ be a closed 3-form. Prove that $H^i(\Lambda^*(M), d_\theta)$ are isomorphic to $H^i(M)$, or find a counterexample.
11.3 Pullback of a differential form

**Definition 11.9.** Let $\phi: M \to N$ be a morphism of smooth manifolds, and $\Lambda^1 N \xrightarrow{\phi^*} \Lambda^1 M$ an induced morphism which maps $f dg$ to $\phi^* f d\phi^* g$. A form $\phi^* \alpha$ is called **pullback of** $\alpha$.

**Exercise 11.23.** Prove that $\phi^*$ can be extended from $\Lambda^1 N$ to a multiplicative homomorphism

$$\phi^*: \Lambda^* N \to \Lambda^* M.$$ 

**Definition 11.10.** Pullback of an $i$-form $\alpha$ is a form $\phi^* \alpha$ defined as above. If $M \xrightarrow{\phi} N$ is a closed embedding, the form $\phi^* \alpha$ is called **restriction of** $\alpha$ to $M \hookrightarrow N$.

**Exercise 11.24.** Let $x \in T_m M$ be a tangent vector, and $\alpha \in \Lambda^1 N$ a 1-form. Prove that $\phi^* \alpha(x) = \alpha(D_\phi(x))$, where $D_\phi: T_m M \to T_{\phi(m)} N$ is a differential.

**Exercise 11.25 (!).** Prove that $\phi^* d\alpha = d\phi^* \alpha$.

**Hint.** Use exercise 11.8.

**Definition 11.11.** Let $f: M \times [a, b] \to M$ be a smooth map such that for all $t \in [a, b]$ the restriction $f_t := f \big|_{M \times \{t\}}: M \to M$ is a diffeomorphism. Then $f$ is called a **flow of diffeomorphisms**.

**Exercise 11.26.** Let $V_t$ be a flow of diffeomorphisms, $f \in C^\infty M$, and $V^*_t(f)(x) := f(V_t(x))$. Consider the map $\frac{d}{dt} V_t \big|_{t=c}: C^\infty M \to C^\infty M$, with $\frac{d}{dt} V_t \big|_{t=c} (f) = (V^{-1}_c)^* \frac{d}{dt} f \big|_{t=c}$. Show that $\frac{d}{dt} V_t \big|_{t=c}$ is a derivation (that is, a vector field).

**Definition 11.12.** The vector field $\frac{d}{dt} V_t \big|_{t=c}$ is called a **vector field tangent to** a flow of diffeomorphisms $V_t$ at $t = c$.

**Definition 11.13.** Let $v$ be a vector field on $M$, and $V: M \times [a, b] \to M$ a flow of diffeomorphisms which satisfies $\frac{d}{dt} V_t \big|_{t=c} = v$ for each $c$, and $V_0 = \text{id}$. Then $V_t$ is called an **exponent of** $v$.

**Exercise 11.27.** Prove that an exponent of a vector field is unique, if it exists.

**Exercise 11.28 (*).** Let $v$ be a vector field, and $D_v: C^\infty M \to C^\infty M$ the corresponding derivation. Suppose that the serie $e^{t D_v}(f) := \sum_{i=0}^{\infty} \frac{t^i}{i!} (D_v)^i f$ converges for some $f \in C^\infty M$ and $t \in [0, a]$, and the exponent $V_t$ of $v$ exists for all $t \in [0, a]$. Prove that $e^{t D_v}(f) = V^*_t(f)$.

**Exercise 11.29 (!).** Let $v$ be a vector field, and $V_t$ its exponent. For any $\alpha \in \Lambda^* M$, consider $V^*_t \alpha$ as a $\Lambda^* M$-valued function of $t$. Prove that $\text{Lie}_v \alpha = \frac{d}{dt} (V^*_t \alpha)$.

**Hint.** Use exercise 11.8.
11.4 Computation of cohomology

Exercise 11.30 (*). Let $M$ be a smooth manifold equipped with a smooth action $X_t$ of the group $S^1 = U(1)$, $f \in C^\infty M$, and $f_t := X_t^* f$. Prove that $f_t = \sum_{n \in \mathbb{Z}} e^{2\pi \sqrt{-1}nt} f_t$, where $f_t \in C^\infty M$ are $S^1$-invariant functions.

Definition 11.14. Let $A \subset \Lambda^* M$ be a subspace which satisfies $d(A) \subset A$. Cohomology of $A$ is the quotient $H^*(A) := \ker \frac{d}{d(A)}$. If $A$ is graded, $A = \bigoplus A^i$, where $A_i := \Lambda^i(M) \cap A$, one has $H^*(A) = \bigoplus_i H^i(A)$, where $H^i(A) = \frac{\ker d|_{A^i}}{d(A^i)}$.

Exercise 11.31. Let $M$ be a smooth manifold, and $v \in TM$ a vector field. Prove that the image $\text{Lie}_v \Lambda^* M$ satisfies $d(\text{Lie}_v \Lambda^* M) \subset \text{Lie}_v \Lambda^* M$, and $H^*(\text{Lie}_v \Lambda^* M) = 0$.

Exercise 11.32. Let $M$ be a smooth manifold, and $v \in TM$ a vector field. Consider the space $\Lambda^*_v M := \ker \text{Lie}_v |_{\Lambda^* M}$. Let $V_t$ be an exponent of $v$. Suppose that $V_t$ is periodic, $V_{t+2\pi} = V_t$.

a. (!) Prove that $(\text{Lie}_v \Lambda^* M) \cap \Lambda^*_v M = \emptyset$.

b. (*) Prove that $(\text{Lie}_v \Lambda^* M) \oplus \Lambda^*_v M = \Lambda^* M$.

Hint. Use exercise 11.30.

Exercise 11.33. Let $M$ be a smooth manifold, and $v \in TM$ a vector field. Suppose that $(\text{Lie}_v \Lambda^* M) \oplus \Lambda^*_v M = \Lambda^* M$. Prove that $H^*(M) = H^*(\Lambda^*_v M)$.

Exercise 11.34. Let $M$ be a smooth manifold, $K \subset \mathbb{R}^n$ a starlike set, and $v$ the radial vector field on $K$ lifted to $M \times K$.

a. (!) Prove that $\Lambda^*_v (M \times K) = \Lambda^* (M)$.

b. Prove that $H^*(M \times K) = H^*(M)$.

Exercise 11.35. Let $v_1, v_2$ be vector fields which satisfy $\text{Lie}_{v_1} \Lambda^* M \oplus \Lambda^* M v_1 = \Lambda^* M$.

a. Prove that $\Lambda^* M = (\ker \text{Lie}_{v_1} \cap \ker \text{Lie}_{v_2}) \oplus (\text{im} \text{Lie}_{v_1} \cup \text{im} \text{Lie}_{v_2})$.

b. (!) Prove that $H^*(\text{im} \text{Lie}_{v_1} \cup \text{im} \text{Lie}_{v_2}) = 0$.

Exercise 11.36 (*). Let $G$ be a Lie group (that is, a smooth manifold equipped with a smooth group structure) generated by circles $S^1 \subset G$, $M$ a manifold with $G$-action, and $\Lambda^*_G M$ the space of $G$-invariant forms. Prove that $H^*(\Lambda^*_G M) = H^* M$.

Hint. Use the previous exercise.

Exercise 11.37 (*). Compute cohomology of a sphere $S^n$ and a real projective space $\mathbb{R}P^n$.

Hint. Use the previous exercise.