Geometry 2: Remedial topology

Rules: You may choose to solve only "hard" exercises (marked with !, * and **) or "ordinary" ones (marked with ! or unmarked), or both, if you want to have extra problems. To have a perfect score, a student must obtain (in average) a score of 10 points per week. It's up to you to ignore handouts entirely, because passing tests in class and having good scores at final exams could compensate (at least, partially) for the points obtained by grading handouts.

Solutions for the problems are to be explained to the examiners orally in the class and marked in the score sheet. It's better to have a written version of your solution with you. It's OK to share your solutions with other students, and use books, Google search and Wikipedia, we encourage it. The first score sheet will be distributed February 11-th.

If you have got credit for 2/3 of ordinary problems or 2/3 of "hard" problems, you receive $6t$ points, where t is a number depending on the date when it is done. Passing all "hard" or all "ordinary" problems (except at most 2) brings you $10t$ points. Solving of "**" (extra hard) problems is not obligatory, but each such problem gives you a credit for 2 "*" or "!" problems in the "hard" set.

The first 3 weeks after giving a handout, $t = 1.5$, between 21 and 35 days, $t = 1$, and afterwards, $t = 0.7$. The scores are not cumulative, only the best score for each handout counts.

Please keep your score sheets until the final evaluation is given.

2.1 Topological spaces

Definition 2.1. A set of all subsets of M is denoted 2^M. **Topology** on M is a collection of subsets $S \subset 2^M$ called **open subsets**, and satisfying the following conditions.

1. Empty set and M are open
2. A union of any number of open sets is open
3. An intersection of a finite number of open subsets is closed.

A complement of an open set is called **closed**. A set with topology on it is called **a topological space**. **An open neighbourhood** of a point is an open set containing this point.

Definition 2.2. A map $\phi: M \longrightarrow M'$ of topological spaces is called **continuous** if a preimage of each open set $U \subset M'$ is open in M. A bijective continuous map is called **a homeomorphism** if its inverse is also continuos.

Exercise 2.1. Let M be a set, and S a set of all subsets of M. Prove that S defines topology on M. This topology is called **discrete**. Describe the set of all continuous maps from M to a given topological space.

Exercise 2.2. Let M be a set, and $S \subset 2^M$ a set of two subsets: empty set and M. Prove that S defines topology on M. This topology is called **codiscrete**. Describe the set of all continuous maps from M to a space with discrete topology.

Definition 2.3. Let M be a topological space, and $Z \subset M$ its subset. **Open subsets** of Z are subsets obtained as $Z \cap U$, where U is open in M. This topology is called **induced topology**.

Definition 2.4. A metric space is a set M equipped with a **distance function** $d : M \times M \longrightarrow \mathbb{R}^{\geqslant 0}$ satisfying the following axioms.

1. $d(x, y) = 0$ iff $x = y$.
2. $d(x, y) = d(y, x)$.
3. (triangle inequality) $d(x, y) + d(y, z) \geqslant d(x, z)$.

An **open ball** of radius r with center in x is $\{y \in M \mid d(x, y) < r\}$.

Definition 2.5. Let M be a metric space. A subset $U \subset M$ is called **open** if it is obtained as a union of open balls. This topology is called **induced by the metric**.

Definition 2.6. A topological space is called **metrizable** if its topology can be induced by a metric.

Exercise 2.3. Show that discrete topology can be induced by a metric, and codiscrete cannot.

Exercise 2.4. Prove that an intersection of any collection of closed subsets of a topological space is closed.

Definition 2.7. An intersection of all closed supersets of $Z \subset M$ is called **closure** of Z.

Definition 2.8. A limit point of a set $Z \subset M$ is a point $x \in M$ such that any neighbourhood of M contains a point of Z other than x. **A limit** of a sequence $\{x_i\}$ of points in M is a point $x \in M$ such that any neighbourhood of $x \in M$ contains all x_i for all i except a finite number. A sequence which has a limit is called **convergent**.

Exercise 2.5. Show that a closure of a set $Z \subset M$ is a union of Z and all its limit points.

Exercise 2.6. Let $f : M \longrightarrow M'$ be a continuous map of topological spaces. Prove that $f(\lim_i x_i) = \lim_i f(x_i)$ for any convergent sequence $\{x_i \in M\}$.

Exercise 2.7. Let $f: M \longrightarrow M'$ be a map of metrizable topological spaces, such that $f(\lim_i x_i) = \lim_i f(x_i)$ for any convergent sequence $\{x_i \in M\}$. Prove that f is continuous.

Exercise 2.8 (*). Find a counterexample to the previous problem for non-metrizable, Hausdorff topological spaces.

Exercise 2.9 (**). Let $f: M \longrightarrow M'$ be a map of countable topological spaces, such that $f(\lim_i x_i) = f(\lim_i x_i)$ for any convergent sequence $\{x_i \in M\}$. Prove that f is continuous, or find a counterexample.

Exercise 2.10 (*). Let $f: M \longrightarrow N$ be a bijective map inducing homeomorphisms on all countable subsets of M. Show that it is a homeomorphism, or find a counterexample.

2.2 Hausdorff spaces

Definition 2.9. Let M be a topological space. It is called **Hausdorff**, or **separable**, if any two distinct points $x \neq y \in M$ can be **separated** by open subsets, that is, there exist open neighbourhoods $U \ni x$ and $V \in y$ such that $U \cap V = \emptyset$.

Remark 2.1. In topology, the Hausdorff axiom is usually assumed by default. In subsequent handouts, it will be always assumed (unless stated otherwise).

Exercise 2.11. Prove that any subspace of a Hausdorff space with induced topology is Hausdorff.

Exercise 2.12. Let M be a Hausdorff topological space. Prove that all points in M are closed subsets.

Exercise 2.13. Let M be a topological space, with all points of M closed. Prove that M is Hausdorff, or find a counterexample.

Exercise 2.14. Count the number of non-isomorphic topologies on a finite set of 4 elements. How many of these topologies are Hausdorff?

Exercise 2.15 (!). Let Z_1, Z_2 be non-intersecting closed subsets of a metrizable space M. Find open subsets $U \supset Z_1$, $V \supset Z_2$ which do not intersect.

Definition 2.10. Let M, N be topological spaces. **Product topology** is a topology on $M \times N$, with open sets obtained as a union of $U \times V$, where U is open in M and V is open in N.

Exercise 2.16. Prove that a topology on X is Hausdorff if and only if the diagonal $\{(x, y) \in X \times X \ \mid \ x = y\}$ is closed in the product topology.

Definition 2.11. Let $\sim$ be an equivalence relation on a topological space M. **Factor-topology** (or **quotient topology**) is a topology on the set $M/\sim$ of equivalence classes such that a subset $U \subset M/\sim$ is open whenever its preimage in M is open.

Exercise 2.17. Let G be a finite group acting on a Hausdorff topological space M.[1] Prove that the quotient map is closed.[2]

Exercise 2.18 (*). Let $\sim$ be an equivalence relation on a topological space M, and $\Gamma \subset M \times M$ **its graph**, that is, the set $\{(x, y) \in M \times M \ \mid \ x \sim y\}$. Suppose that the map $M \longrightarrow M/\sim$ is open, and the Γ is closed in $M \times M$. Show that $M/\sim$ is Hausdorff.

Hint. Prove that diagonal is closed in $M \times M$.

Exercise 2.19 (!). Let G be a finite group acting on a Hausdorff topological space M. Prove that M/G with the quotient topology is Hausdorff.

Hint. Use the previous exercise.

Exercise 2.20 (**). Let $M = \mathbb{R}$, and $\sim$ an equivalence relation with at most 2 elements in each equivalence class. Prove that $\mathbb{R}/\sim$ is Hausdorff, or find a counterexample.

Exercise 2.21 (*). ("gluing of closed subsets") Let M be a metrizable topological space, and $Z_i \subset M$ a finite number closed subsets which do not intersect, grouped into pairs of homeomorphic $Z_i \sim Z_i'$. Let $\sim$ an equivalence relation generated by these homeomorphisms. Show that $M/\sim$ is Hausdorff.

[1] Speaking of a group acting on a topological space, one always means continuous action.

[2] a **closed map** is a map which puts closed subsets to closed subsets.

2.3 Compact spaces

Definition 2.12. **A cover** of a topological space M is a collection of open subsets $\{U_\alpha \in 2^M\}$ such that $\bigcup U_\alpha = M$. **A subcover** of a cover $\{U_\alpha\}$ is a subset $\{U_\beta\} \subset \{U_\alpha\}$. A topological space is called **compact** if any cover of this space has a finite subcover.

Exercise 2.22. Let M be a compact topological space, and $Z \subset M$ a closed subset. Show that Z is also compact.

Exercise 2.23. Let M be a countable, metrizable topological space. Show that either M contains a converging sequence of pairwise different elements, or M contains a subset with discrete topology.

Definition 2.13. A topological space is called **sequentially compact** if any sequence $\{z_i\}$ of points of M has a converging subsequence.

Exercise 2.24. Let M be metrizable a compact topological space. Show that M is sequentially compact.

Hint. Use the previous exercise.

Remark 2.2. **Heine-Borel theorem** says that the converse is also true: any metric space which is sequentially compact, is also compact. Its proof is moderately difficult (please check Wikipedia or any textbook on point-set topology, metric geometry or analysis; "Metric geometry" by Burago-Burago-Ivanov is probably the best place).

In subsequent handouts, you are allowed to use this theorem without a proof.

Exercise 2.25 (*). Construct an example of a Hausdorff topological space which is sequentially compact, but not compact.

Exercise 2.26 (*). Construct an example of a Hausdorff topological space which is compact, but not sequentially compact.

Definition 2.14. A **topological group** is a topological space with group operations $G \times G \longrightarrow G$, $x, y \mapsto xy$ and $G \longrightarrow G$, $x \mapsto x^{-1}$ which are continuous. In a similar way, one defines **topological vector spaces**, **topological rings** and so on.

Exercise 2.27 (*). Let G be a compact topological group, acting on a topological space M in such a way that the map $M \times G \longrightarrow M$ is continuous. Prove that the quotient space is Hausdorff.

Exercise 2.28. Let $f: X \longrightarrow Y$ be a continuous map of topological spaces, with X compact. Prove that $f(X)$ is also compact.

Exercise 2.29. Let $Z \subset Y$ be a compact subset of a Hausdorff topological space. Prove that it is closed.

Exercise 2.30. Let $f: X \longrightarrow Y$ be a continuous, bijective map of topological spaces, with X compact and Y Hausdorff. Prove that it is a homeomorphism.

Definition 2.15. A topological space M is called **pseudocompact** if any continuous function $f: M \longrightarrow \mathbb{R}$ is bounded.

Exercise 2.31. Prove that any compact topological space is pseudocompact.

Hint. Use the previous exercise.

Exercise 2.32. Show that for any continuous function $f: M \longrightarrow \mathbb{R}$ on a compact space there exists $x \in M$ such that $f(x) = \sup_{z \in M} f(z)$.

Exercise 2.33. Consider $\mathbb{R}^n$ as a metric space, with the standard (Euclidean) metric. Let $Z \subset \mathbb{R}^n$ be a closed, bounded set ("bounded" means "contained in a ball of finite radius"). Prove that Z is sequentially compact.

Exercise 2.34 (**). Find a pseudocompact Hausdorff topological space which is not compact.

Definition 2.16. A map of topological spaces is called **proper** if a preimage of any compact subset is always compact.

Exercise 2.35 (*). Let $f: X \longrightarrow Y$ be a continuous, proper, bijective map of metrizable topological spaces. Prove that f is a homeomorphism, or find a counterexample.

Exercise 2.36 (*). Let $f: X \longrightarrow Y$ be a continuous, proper map of metrizable topological spaces. Show that f is closed, or find a counterexample.