Geometry 3: Hausdorff dimension

Rules: You may choose to solve only "hard" exercises (marked with !, * and **) or "ordinary" ones (marked with ! or unmarked), or both, if you want to have extra problems. To have a perfect score, a student must obtain (in average) a score of 10 points per week. It's up to you to ignore handouts entirely, because passing tests in class and having good scores at final exams could compensate (at least, partially) for the points obtained by grading handouts.

Solutions for the problems are to be explained to the examiners orally in the class and marked in the score sheet. It's better to have a written version of your solution with you. It's OK to share your solutions with other students, and use books, Google search and Wikipedia, we encourage it. The first score sheet will be distributed February 11-th.

If you have got credit for 2/3 of ordinary problems or 2/3 of "hard" problems, you receive $6t$ points, where t is a number depending on the date when it is done. Passing all "hard" or all "ordinary" problems (except at most 2) brings you $10t$ points. Solving of "**" (extra hard) problems is not obligatory, but each such problem gives you a credit for 2 "*" or "!" problems in the "hard" set.

The first 3 weeks after giving a handout, $t = 1.5$, between 21 and 35 days, $t = 1$, and afterwards, $t = 0.7$. The scores are not cumulative, only the best score for each handout counts.

Please keep your score sheets until the final evaluation is given.

The original English translation of this handout was done by Sasha Anan′in (UNICAMP) in 2010.

3.1 Hausdorff dimension and measure

Definition 3.1. Let M be a metric space. The **diameter** $\operatorname{diam} M \in [0, \infty]$ is the number $\sup\limits_{x,y\in M} d(x, y)$.

Definition 3.2. In a metric space, **a ball** of radius ε centered at x is defined as the set of all points y satisfying $d(x, y) < \varepsilon$.

Exercise 3.1. Describe all possible values of the diameter of the ball of radius ε in a metric space.

Exercise 3.2. Let M be a metric space and let $\varepsilon > 0$. Show that M admits a cover by balls of diameter $\leq \varepsilon$.

Definition 3.3. Let $\{S_i\}$ be a cover of a metric space M by balls of radius r with $r < \varepsilon$. Define $\mu_{d,\varepsilon} \in [0, \infty]$ as

$$\mu_{d,\varepsilon} M := \inf_{\{S_i\}} \sum_i (\operatorname{diam} S_i)^d,$$

where the infimum is taken over all such covers. The limit

$$\mu_d M := \sup \lim_{\varepsilon\to 0} \mu_{d,\varepsilon} M$$

is called d**-dimensional Hausdorff measure** of M.

Exercise 3.3. Consider $M = \mathbb{R}^n$ with a metric given by the norm

$$\big|(x_1, \dots, x_n)\big| := \max |x_i|.$$

Show that the n-dimensional Hausdorff measure of a polyhedron equals its volume (in the usual sense).

Exercise 3.4. Consider the metric on $M = \mathbb{R}^n$ given by the norm $\big|(x_1, \dots, x_n)\big| := \sum |x_i|$.

a. Prove that the n-dimensional Hausdorff measure of a polyhedron is proportional to its volume.

b. (*) Calculate the coefficient of proportionality.

Exercise 3.5. Consider $M = \mathbb{R}^n$ with the usual (Euclidean) metric.

a. Show that the n-dimensional Hausdorff measure of a polyhedron is proportional to its volume.

b. (*) Calculate the coefficient of proportionality.

Definition 3.4. A map $f : M \to N$ of metric spaces is called Lipschitz with constant $C > 0$ if $d(x, y) \geq C \cdot d\big(f(x), f(y)\big)$ for all $x, y \in M$. A map is called bi-Lipschitz if it is bijective and the inverse map is also Lipschitz (with some constant).

Exercise 3.6. Show that every Lipschitz map is continuous.

Exercise 3.7 (*). Construct an example of a continuous map of metric spaces that is not Lipschitz.

Exercise 3.8. Let d_1, d_2 be two norms on a vector space V. Denote the corresponding metrics by the same letters. Prove that the identity map $\mathsf{Id}_V : (V, d_1) \longrightarrow (V, d_2)$ is Lipschitz if and only if the unit ball $B_1(r, d_1)$ is bounded in the metric d_2.

Exercise 3.9 (*). Let $M = \mathbb{R}^n$ and let d_1, d_2 be some norms on M. Show that $\mathsf{Id}_M : (M, d_1) \to (M, d_2)$ is bi-Lipschitz.

Exercise 3.10 (!). Let $U \subset \mathbb{R}^n$ be a bounded open subset and $\Phi : U \to \mathbb{R}^n$ a smooth map which can be smoothly extended to the boundary ∂U. Prove that Φ is Lipschitz.

Exercise 3.11 (!). Let $M \xrightarrow{f} N$ be a Lipschitz map of metric spaces with constant C. Show that $\mu_d M \geq C^d \mu_d f(M)$, where μ_d is d-dimensional Hausdorff measure on M.

Exercise 3.12 (!). Suppose that $\mu_d M < \infty$. Show that $\mu_{d'} M = 0$ for every $d' > d$.

Hint. Deduce from $\operatorname{\mathsf{diam}} S_i < \varepsilon$ the inequality

$$\mu_{d',\varepsilon} M = \inf_{\{S_i\}} \sum_i (\operatorname{\mathsf{diam}} S_i)^{d'} \leq \varepsilon^{d'-d} \inf_{\{S_i\}} \sum_i (\operatorname{\mathsf{diam}} S_i)^d = \varepsilon^{d'-d} \mu_{d,\varepsilon} M \quad (3.1)$$

and pass to the limit $\varepsilon \to 0$.

Exercise 3.13 (!). Suppose that $\mu_{d'} M = \infty$. Show that $\mu_d M = \infty$ for every $d < d'$.

Hint. Use the inequality (3.1) and pass to the limit $\varepsilon \to 0$.

Definition 3.5. Let M be a metric space. The **Hausdorff dimension** $\dim_H M \in [0, \infty]$ is the supremum of all d such that $\mu_d M = \infty$.

Exercise 3.14. Find the Hausdorff dimension of a finite set.

Exercise 3.15. Let $f : M \to N$ be a Lipschitz map. Show that f does not increase the Hausdorff dimension: $\dim_H M \geq \dim_H f(M)$.

Exercise 3.16. Show that every bi-Lipschitz map preserve Hausdorff dimension ("Hausdorff dimension is a bi-Lipschitz invariant").

Exercise 3.17 (*). Find the Hausdorff dimension of the Cantor set $K \subset [0, 1]$, obtained as a set of all real numbers without a number 1 in their ternary expansion.

Definition 3.6. A subset $Z \subset \mathbb{R}^n$ has **measure zero** if for every $\varepsilon > 0$ there exists a countable cover of Z by balls U_i such that $\sum_i \operatorname{Vol} U_i < \varepsilon$.

Exercise 3.18. Show that the countable union of subsets of zero measure has measure zero.

Exercise 3.19. Show that the image of a subset of zero measure under a Lipschitz map $\mathbb{R}^n \to \mathbb{R}^n$ has measure zero.

Exercise 3.20 (!). Show that the image of a subset of zero measure under a smooth map $\mathbb{R}^n \to \mathbb{R}^n$ has measure zero.

Hint. Prove that a smooth map is Lipschitz, and use the previous exercise.

Exercise 3.21 (*). Construct an example of a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^n$ that sends a subset of zero measure to a subset of nonzero measure.

Exercise 3.22 (!). Let $M \subset \mathbb{R}^d$ be a subset such that $\dim_H M < d$. Show that M has measure zero.

Definition 3.7. Let M be a smooth manifold with a countable atlas $\{U_i,\ \varphi_i : U_i \to \mathbb{R}^n\}$. A subset $Z \subset M$ has **measure zero** if the image $\varphi(Z \cap U_i)$ has measure zero in $\mathbb{R}^n$ for every i.

Exercise 3.23. Show that this definition does not depend on the choice of an atlas on M.

Exercise 3.24. Let $M \xrightarrow{f} \mathbb{R}^n$ be a smooth map of manifolds and let M be a union of compact subsets. Show that $\dim_H f(M) \leq \dim M$.

Hint. Show first that f is Lipschitz on compact subsets. Then use the fact that Lipschitz maps satisfy $\dim_H f(M) \leq \dim M$.

Exercise 3.25 (!). Let $M \xrightarrow{f} N$ be a smooth map of manifolds such that $\dim M < \dim N$. Show that the image of M has measure zero.

Hint. Use the previous exercise.

Remark 3.1. This theorem is a special case of **Sard's lemma** that claims that the set of critical values of a smooth map has measure zero.

3.2 Whitney's theorem (with a bound on dimension

Definition 3.8. The **Klein bottle** is the quotient of the two-dimensional torus $T^2 := \mathbb{S}^1 \times \mathbb{S}^1$ by the action of the group $\mathbb{Z}/2\mathbb{Z}$ mapping (t_1, t_2) to $(t_1 + \pi, -t_2)$.

Exercise 3.26. Show that this action is free, and the quotient is a manifold.

Exercise 3.27. Let $M \xrightarrow{f} N$ be a smooth map of manifolds, $f(x) = y$, and $U \ni x$, $V \ni y$ charts, equipped with the embeddings $U \hookrightarrow \mathbb{R}^m$, $V \hookrightarrow \mathbb{R}^n$. Choose U and V in such a way that $f(U) \subset V$, and consider $f\big|_U$ as a map from $U \subset \mathbb{R}^m$ to $\mathbb{R}^n$. Suppose that the differential $Df: T_x\mathbb{R}^m \longrightarrow T_y\mathbb{R}^n$ is injective for one choice of the charts U, V. Prove that it is injective for any other choice of the charts.

Definition 3.9. A smooth map of manifolds $M \xrightarrow{f} N$ is called **immersion** if its differential $Df: T_xM \longrightarrow T_{f(x)}N$, computed in local coordinates, is injective.

Exercise 3.28. Construct an immersion of the Klein bottle into $\mathbb{R}^3$.

Exercise 3.29 (!). Let $M \xrightarrow{f} N$ be a smooth map of manifolds, where M is compact. Show that f is a smooth embedding if and only if it is an injective immersion.

Hint. Use the inverse function theorem.

Definition 3.10. Let $M \hookrightarrow \mathbb{R}^n$ be a smooth m-submanifold. The **tangent plane** at $p \in M$ is the plane in $\mathbb{R}^n$ tangent to M (i.e, the plane lying in the image of the differential given in local coordinates). A **tangent vector** is an arbitrary vector in this plane with the origin at p. The space of all tangent vectors at p is denoted by T_pM. Given a metric on $\mathbb{R}^n$, we can define the space of **unit tangent vectors** $\mathbb{S}^{m-1}M$ as the set of all pairs (p, v), where $p \in M$, $v \in T_pM$, and $|v| = 1$.

Exercise 3.30. Prove that $\mathbb{S}^{m-1}M$ is a smooth manifold and that the natural projection $\mathbb{S}^{m-1}M \to M$ is a smooth map with fibers $\mathbb{S}^{m-1}$.

Remark 3.2. $\mathbb{S}^{m-1}M$ is called the **unit sphere bundle** over M.

Exercise 3.31 (*). Show that the manifold $\mathbb{S}^{m-1}M$ does not depend on an embedding $M \hookrightarrow \mathbb{R}^n$, i.e., for any two embeddings of M into $\mathbb{R}^n$ and into $\mathbb{R}^{n'}$, the corresponding manifolds $\mathbb{S}^{m-1}M$ are diffeomorphic.

Exercise 3.32 (!). Let $M \xhookrightarrow{\varphi} \mathbb{R}^n$ be a manifold of dimension m embedded into $\mathbb{R}^n$, $\lambda \in \mathbb{P}^{n-1}_{\mathbb{R}}$ a straight line in $\mathbb{R}^n$, and let $P_\lambda : \mathbb{R}^n \to \mathbb{R}^{n-1}$ denote the projection onto the quotient $\mathbb{R}^n/\lambda \cong \mathbb{R}^{n-1}$.

a. Denote the diagonal by $\Delta \subset M \times M$. Define the map $M \times M \setminus \Delta \xrightarrow{B} \mathbb{P}^{n-1}_{\mathbb{R}}$ by sending the pair of points $(x, y) \in M \times M$ to the straight line passing through $\varphi(x) - \varphi(y)$. Show that $P_\lambda \circ \varphi : M \to \mathbb{R}^{n-1}$ is an injection if and only if λ does not lie in the image of B.

b. Let $\mathbb{S}^{m-1}M \xrightarrow{B_0} \mathbb{P}^{n-1}_{\mathbb{R}}$ be a map sending a tangent vector to the corresponding line in $\mathbb{R}^n$. Show that $P_\lambda \circ \varphi : M \to \mathbb{R}^{n-1}$ is an immersion if and only if λ does not lie in the image of B_0.

Exercise 3.33 (!). Let $M \xhookrightarrow{\varphi} \mathbb{R}^n$ be an embedded manifold of dimension m with $n > 2m + 2$. Show that there exists a projection $\mathbb{R}^n \xrightarrow{P} \mathbb{R}^{2m+2}$ such that $P \circ \varphi : M \to \mathbb{R}^{2m+2}$ is an immersion.

Hint. Use the fact that the images of the maps B_0 and B in the previous problem have measure zero and apply induction on n.

Exercise 3.34. In assumptions of the previous exercise, prove that there exists a projection $\mathbb{R}^n \xrightarrow{P} \mathbb{R}^{2m+1}$ such that $P \circ \varphi : M \to \mathbb{R}^{2m+1}$ is an immersion.

Exercise 3.35. Is any n-dimensional manifold embeddable in $\mathbb{R}^{2n-1}$?

Exercise 3.36 (**). Is it possible to construct an immersion of the complex projective space $\mathbb{P}^2_{\mathbb{C}}$ into $\mathbb{R}^5$?

Exercise 3.37. Let M be a compact manifold of dimension n. Show that M admits a smooth closed embedding into $\mathbb{R}^{2n+2}$.

Remark 3.3. Whitney showed that any m-dimensional manifold with a countable basis of topology admits a closed embedding into $\mathbb{R}^{2m}$. This statement is called the "strong Whitney theorem."