Geometry 5: Vector fields and derivations

Rules: You may choose to solve only "hard" exercises (marked with !, * and **) or "ordinary" ones (marked with ! or unmarked), or both, if you want to have extra problems. To have a perfect score, a student must obtain (in average) a score of 10 points per week. It's up to you to ignore handouts entirely, because passing tests in class and having good scores at final exams could compensate (at least, partially) for the points obtained by grading handouts.

Solutions for the problems are to be explained to the examiners orally in the class and marked in the score sheet. It's better to have a written version of your solution with you. It's OK to share your solutions with other students, and use books, Google search and Wikipedia, we encourage it. The first score sheet will be distributed February 11-th.

If you have got credit for 2/3 of ordinary problems or 2/3 of "hard" problems, you receive $6t$ points, where t is a number depending on the date when it is done. Passing all "hard" or all "ordinary" problems (except at most 2) brings you $10t$ points. Solving of "**" (extra hard) problems is not obligatory, but each such problem gives you a credit for 2 "*" or "!" problems in the "hard" set.

The first 3 weeks after giving a handout, $t = 1.5$, between 21 and 35 days, $t = 1$, and afterwards, $t = 0.7$. The scores are not cumulative, only the best score for each handout counts.

Please keep your score sheets until the final evaluation is given.

5.1 Derivations of a ring

Remark 5.1. All rings in these handouts are assumed to be commutative and with unit. Algebras are associative, but not necessarily commutative (such as the matrix algebra). **Rings over a field** k are rings containing a field k.

Definition 5.1. Let R be a ring over a field k. A k-linear map $D\ R \longrightarrow R$ is called **a derivation** if it satisfies **the Leibnitz equation** $D(fg) = D(f)g+gD(f)$. The space of derivations is denoted as $\operatorname{Der}_k(R)$.

Exercise 5.1. Let $D \in \operatorname{Der}_k(R)$. Prove that $D\Big|_k = 0$.

Exercise 5.2. Let D_1, D_2 be derivations. Prove that the commutator $[D_1, D_2] := D_1D_2 - D_2D_1$ is also a derivation.

Exercise 5.3 (!). Let $K \supset k$ be a field which contains a field k of characteristic 0, and is finite-dimensional over k (such fields K are called **finite extensions** of k). Find the space $\operatorname{Der}_k(K)$.

Exercise 5.4 (*). Is it true if $\operatorname{char} k = p$?

Exercise 5.5. Consider a ring $k[\varepsilon]$, given by a relation $\varepsilon^2 = 0$. Find $\operatorname{Der}_k(k[\varepsilon])$.

Exercise 5.6 (*). Find all rings R over $\mathbb{C}$ such that R is finite-dimensional over $\mathbb{C}$, and $\operatorname{Der}_{\mathbb{C}}(R) = 0$.

Exercise 5.7 (**). Let $D \in \operatorname{Der}_k(K)$ be a derivation of a field K over k, $\operatorname{char} k = 0$, and $[K' : K]$ a finite field extension. Prove that D can be extended to a derivation $D' \in \operatorname{Der}_k(K')$.

Exercise 5.8. Let $D \in \operatorname{Der}_k(R)$ be a derivation, and $I \subset R$ – an ideal. Prove that $D(I^k) \subset I^{k-1}$.

5.2 Modules over a ring

Definition 5.2. Let R be a ring over a field k. **An R-module** is a vector space V over k, equipped with an algebra homomorphism $R \longrightarrow \operatorname{End}(V)$, where $\operatorname{End}(V)$ denotes the endomorphism algebra of V, that is, the matrix algebra.

Exercise 5.9. Let R be a field. Prove that R-modules are the same as vector spaces over R.

Remark 5.2. An R-module is a group, equipped with an operation of "multiplication by elements of R", and satisfying the same axioms of distributivity and associativity as in the definition of a vector space.

Remark 5.3. Homomorphisms, isomorphisms, submodules, quotient modules, direct sums of modules are defined in the same way as for the vector spaces. A ring R is itself an R-module. A direct sum of n copies of R is denoted R^n. Such R-module is called **a free R-module**.

Remark 5.4. R-submodules in R are the same as ideals in R.

Definition 5.3. A ring R is called **a principal ideal ring**, if all non-zero submodules of R are isomorphic to R.

Exercise 5.10. Prove that R is a principal ideal ring iff R has no zero divisors, and all ideals in R are **principal**, that is, are of form Rx, for some non-invertible $x \in R$.

Exercise 5.11. Are these rings principal ideal rings?

a. $R = \mathbb{C}[t]$

b. (!) $R = \mathbb{C}[t_1, t_2]$

c. (*) $R := \mathbb{R}[x, y]/(x^2 + y^2 = -1)$.

Definition 5.4. Finitely generated R-module is a quotient module of R^n.

Exercise 5.12. Find a finitely generated, non-free R-module for $R = \mathbb{C}[t]$.

Definition 5.5. A Noetherian ring is a ring R with all ideals finitely generated as R-modules.

Exercise 5.13 (*). Let R be a Noetherian ring. Prove that any submodule of a finitely generated R-module is finitely generated.

Exercise 5.14. Let R be a ring obtained as a direct limit of the following diagram. The vertices of this diagram are numbered by natural numbers. The corresponding vector spaces are rings $\mathbb{C}[t]$. The arrows $\phi_{k,km}$ of this diagram are going from the k-th vertex to the km-th. The corresponding homomorphisms of polynomial rings $\mathbb{C}[t] \stackrel{\phi_{k,km}}{\longrightarrow} \mathbb{C}[t]$ are determined by the action of $\phi_{k,km}$ on the polynomial generators: $\phi_{k,km}(t) = t^m$. Prove that R is a set of formal linear combinations $a_1 t^{\alpha_1} + a_2 t^{\alpha_2} + ... + a_2 t^{\alpha_2}$, where $a_i \in \mathbb{C}$, and α_i – non-negative rational numbers, with an obvious formula for multiplication.

Exercise 5.15. Consider a ring R defined in the previous exercise. Prove that an ideal generated by the polynomials $a_1 t^{\alpha_1} + a_2 t^{\alpha_2} + ... + a_2 t^{\alpha_2}$, where all α_i are positive, is not finitely generated.

Exercise 5.16. Consider a ring of germs of smooth functions in a point, and let K be an ideal of all functions with all derivatives of all orders vanishing. Show that this ideal is not principal.

Exercise 5.17 (*). Prove that K is not finitely generated.

Exercise 5.18 (*). Let I be a finitely generated ideal in the ring R of germs of smooth functions on $\mathbb{R}$ at 0. Prove that I is principal, or find a counterexample.

5.3 Vector fields

Remark 5.5. Let R be a ring over k. The space $\operatorname{Der}_k(R)$ of derivations is also an R-module, with multiplicative action of R given by $rD(f) = rD(f)$.

Exercise 5.19. Let $R = k[t_1, .., t_k]$ be a polynomial ring. Prove that $\operatorname{Der}_k(R)$ is a free R-module isomorphic to R^n, with generators $\frac{d}{dt_1}, \frac{d}{dt_2}, ..., \frac{d}{dt_n}$.

Hint. Construct a map $\operatorname{Der}_k(R) \longrightarrow R^n$,

$$D \longrightarrow (D(t_1), D(t_2), ..., D(t_n))$$

and prove that it is an isomorphism of R-modules.

Exercise 5.20 (*). Let $R = k(t_1, .., t_k)$ be a ring of rational functions, that is, the ring of functions $\frac{P}{Q}$, where P and $Q \in k(t_1, .., t_k)$ are arbitrary polynomials, $Q \neq 0$. Prove that $\operatorname{Der}_k(R)$ is a free R-module, isomorphic to R^n.

Exercise 5.21 (!). Prove the **Hadamard's lemma**: Let f be a smooth function f on $\mathbb{R}^n$, and x_i the coordinate functions. Then $f(x) = f(0) + \sum_{i=1}^n x_i g_i(x)$, for some smooth $g_i \in C^\infty\mathbb{R}^n$.

Hint. Consider a function $h(t) \in C^\infty\mathbb{R}^n$, $h(t) = f(tx)$. Then $\frac{dh}{dt} = \sum \frac{df(tx)}{dx_i}(tx)x_i$. Integrating this expression over t, obtain $f(x) - f(0) = \sum_i x_i \int_0^1 \frac{df(tx)}{dx_i}(tx)dt$.

Definition 5.6. Consider coordinates $t_1, ..., t_n$ on $\mathbb{R}^n$, and let

$$\operatorname{Der}(C^\infty\mathbb{R}^n) \xrightarrow{\Pi} (C^\infty\mathbb{R}^n)^n,$$

map D to $(D(t_1), D(t_2), ..., D(t_n))$.

Exercise 5.22. Prove that Π is surjective.

Exercise 5.23. Prove that $\Pi(D) = 0 \Leftrightarrow D(P) = 0$ for each $P(t_1, ..., t_n)$.

Exercise 5.24. Let $\mathfrak{m}_x \subset C^\infty\mathbb{R}^n$ be an ideal of all smooth functions vanishing at $x \in \mathbb{R}^n$. Prove that it is maximal.

Exercise 5.25. Let f be a smooth function on $\mathbb{R}^n$ satisfying $f(x) = 0$ and $f'(x) = 0$. Prove that $f \in \mathfrak{m}_x^2$.

Hint. Use the Hadamard's Lemma.

Exercise 5.26 (!). Let $D \in \operatorname{Der}_{\mathbb{R}}(C^\infty\mathbb{R}^n)$ be a derivation, satisfying $D \in \ker\Pi$ (that is, vanishing on coordinate functions). Prove that for all $f \in C^\infty\mathbb{R}^n$, and all $x \in \mathbb{R}^n$, one has $D(f) \in \mathfrak{m}_x$.

Hint. Use the previous exercise and Exercise 5.8.

Exercise 5.27 (!). Prove that the map

$$\operatorname{Der}(C^\infty\mathbb{R}^n) \xrightarrow{\Pi} (C^\infty\mathbb{R}^n)^n$$

is an isomorphism

Hint. Use the previous exercise.

Exercise 5.28 (**). Find a non-trivial element $\gamma \in \operatorname{Der}_{\mathbb{R}}(C^0\mathbb{R})$ in the space of derivations of continuous functions, or prove that it is empty.

Exercise 5.29 (**). Find a non-trivial element $\gamma \in \operatorname{Der}_{\mathbb{R}}(C^1\mathbb{R})$ in the space of derivations of the ring of differentiable functions of class C^1, or prove that it is empty.