Geometry 6: Vector bundles and sheaves

Rules: You may choose to solve only "hard" exercises (marked with !, * and **) or "ordinary" ones (marked with ! or unmarked), or both, if you want to have extra problems. To have a perfect score, a student must obtain (in average) a score of 10 points per week. It's up to you to ignore handouts entirely, because passing tests in class and having good scores at final exams could compensate (at least, partially) for the points obtained by grading handouts.

Solutions for the problems are to be explained to the examiners orally in the class and marked in the score sheet. It's better to have a written version of your solution with you. It's OK to share your solutions with other students, and use books, Google search and Wikipedia, we encourage it. The first score sheet will be distributed February 11-th.

If you have got credit for 2/3 of ordinary problems or 2/3 of "hard" problems, you receive $6t$ points, where t is a number depending on the date when it is done. Passing all "hard" or all "ordinary" problems (except at most 2) brings you $10t$ points. Solving of "**" (extra hard) problems is not obligatory, but each such problem gives you a credit for 2 "*" or "!" problems in the "hard" set.

The first 3 weeks after giving a handout, $t = 1.5$, between 21 and 35 days, $t = 1$, and afterwards, $t = 0.7$. The scores are not cumulative, only the best score for each handout counts.

Please keep your score sheets until the final evaluation is given.

6.1 Sheaves of modules.

Remark 6.1. Now I will give a new definition of a sheaf. The old definition ("sheaf of functions") becomes a special case of this one.

Definition 6.1. Let M be a topological space. **A sheaf** $\mathcal{F}$ on M is a collection of vector spaces $\mathcal{F}(U)$ defined for each open subset $U\subset M$, with the **restriction maps**, which are linear homomorphisms $\mathcal{F}(U) \stackrel{\phi_{U,U'}}{\longrightarrow} \mathcal{F}(U')$, defined for each $U'\subset U$, and satisfying the following conditions.

(A) Composition of restrictions is again a restriction: for any open subsets $U_1\subset U_2\subset U_3$, the corresponding restriction maps

$$\mathcal{F}(U_1) \stackrel{\phi_{U_1,U_2}}{\longrightarrow} \mathcal{F}(U_2) \stackrel{\phi_{U_2,U_3}}{\longrightarrow} \mathcal{F}(U_3)$$

give $\phi_{U_1,U_2}\circ\phi_{U_2,U_3} = \phi_{U_1,U_3}$.[1]

(B) Let $U\subset M$ be an open subset, and $\{U_i\}$ a cover of U. For any $f\in\mathcal{F}(U)$ such that all restrictions of f to U_i vanish, one has $f = 0$.

(C) Let $U\subset M$ be an open subset, and $\{U_i\}$ a cover of U. Consider a collection $f_i\in\mathcal{F}(U_i)$ of sections, defined for each U_i, and satisfying

$$f_i\Big|_{U_i\cap U_j} = f_j\Big|_{U_i\cap U_j}$$

for each U_i, U_j. Then there exists $f\in\mathcal{F}(U)$ such that the restriction of f to U_i is f_i.

The space $\mathcal{F}(U)$ is called **the space of sections of the sheaf $\mathcal{F}$ on U**. The restriction maps are often denoted $f\longrightarrow f\Big|_U$

Remark 6.2. For a sheaf of functions, the conditions (A) and (B) are satisfied automatically.

[1] If (A) is satisfied, $\mathcal{F}$ is called **a presheaf**.

Exercise 6.1. Let M be a topological space equipped with a presheaf $\mathcal{F}$. Prove that the conditions (B) an (C) are equivalent to exactness of the following sequence.

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$$

for any open $U \subset M$ an open subset, and any cover $\{U_i\}$ of U.

Exercise 6.2. Let $f, g \in C^\infty M$ be functions which are equal on an open subset $U \subset M$, and $D \in \operatorname{Der}_{\mathbb{R}} C^\infty M$ a derivation on a ring of smooth functions. Prove that $D(f)\big|_U = D(g)\big|_U$.

Definition 6.2. Let $U \subset V$ be open subsets in M. We write $U \Subset V$ if the closure of U is contained in V.

Exercise 6.3. Let $U \Subset V$ be open subsets in a smooth metrizable manifold. Prove that there exists a smooth function $\Phi_{U,V} \in C^\infty M$ supported on V and equal to 1 on U.

Exercise 6.4. Let $D \in \operatorname{Der}_{\mathbb{R}} C^\infty M$ be a derivation, and $U \Subset V$ open subsets in M. Given $f \in \mathbb{C}^\infty V$, define $D(f)\big|_U$ using the formula $D(f)\big|_U = D(\Phi_{U,V} \cdot f)$. Prove that $D(f)\big|_U$ satisfies the Leibnitz rule, and is independent from the choice of $\Phi_{U,V}$.

Exercise 6.5 (!). Let $D \in \operatorname{Der}_{\mathbb{R}} C^\infty M$ be a derivation, and $V \subset M$ an open subset in M.

a. Prove that D can be extended to a derivation $D_V \in \operatorname{Der}_{\mathbb{R}} C^\infty V$, in such a way that $D_V\left(f\big|_V\right) = D(f)\big|_V$.

b. Prove that such an extension is unique.

Hint. Use the previous exercise.

Exercise 6.6 (!). Show that this construction makes $\operatorname{Der}_{\mathbb{R}}(C^\infty M)$ into a sheaf of modules over $C^\infty M$.

Definition 6.3. A sheaf homomorphism $\psi: \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ is a collection of homomorphisms

$$\psi_U: \mathcal{F}_1(U) \longrightarrow \mathcal{F}_2(U),$$

defined for each $U \subset M$, and commuting with the restriction maps. **A sheaf isomorphism** is a homomorphism $\Psi: \mathcal{F}_1 \longrightarrow \mathcal{F}_2$, for which there exists an homomorphism $\Phi: \mathcal{F}_2 \longrightarrow \mathcal{F}_1$, such thate $\Phi \circ \Psi = \mathsf{Id}$ and $\Psi \circ \Phi = \mathsf{Id}$.

Exercise 6.7. Let $\psi: \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ be a sheaf homomorphism.

a. Show that $U \longrightarrow \ker \psi_U$ and $U \longrightarrow \operatorname{coker} \psi_U$ are presheaves.

b. Prove that $U \longrightarrow \ker \psi_U$ is a sheaf (it is called **the kernel** of a homomorphism ψ).

c. (*) Prove that $U \longrightarrow \operatorname{coker} \psi_U$ is not always a sheaf (find a counterexample).

Definition 6.4. A subsheaf $\mathcal{F}' \subset \mathcal{F}$ is a sheaf associating to each $U \subset M$ a subspace $\mathcal{F}'(U) \subset \mathcal{F}(U)$.

Exercise 6.8. Find a non-zero sheaf $\mathcal{F}$ on M such that $\mathcal{F}(M) = 0$.

Remark 6.3. Let $A: \ \phi \longrightarrow B$ be a ring homomorphism, and V a B-module. Then V is equipped with a natural A-module structure: $av := \phi(a)v$.

Definition 6.5. A sheaf of rings on a manifold M is a sheaf $\mathcal{F}$ with all the spaces $\mathcal{F}(U)$ equipped with a ring structure, and all restriction maps ring homomorphisms.

Definition 6.6. Let $\mathcal{F}$ be a sheaf of rings on a topological space M, and $\mathcal{B}$ another sheaf. It is called **a sheaf of $\mathcal{F}$-modules** if for all $U \subset M$ the space of sections $\mathcal{B}(U)$ is equipped with a structure of $\mathcal{F}(U)$-module, and for all $U' \subset U$, the restriction map $\mathcal{B}(U) \stackrel{\phi_{U,U'}}{\longrightarrow} \mathcal{B}(U')$ is a homomorphism of $\mathcal{F}(U)$-modules (use Remark 6.3 to obtain a structure of $\mathcal{F}(U)$-module on $\mathcal{B}(U')$).

Exercise 6.9. Let $\mathcal{F}_1$ be a sheaf of rings and $\mathcal{F}$ its subsheaf. Prove that $\mathcal{F}$ is a sheaf of modules over $\mathcal{F}$.

Definition 6.7. The space of germs of a sheaf $\mathcal{F}$ at $x \in M$ is the limit $\underset{\longrightarrow}{\lim}\, \mathcal{F}(U)$, where U is the set of all neighbourhoods of x, and the maps are restriction maps.

Exercise 6.10. Let $\mathcal{F}$ be a ring sheaf on M. Prove that the space of germs of a sheaf of $\mathcal{F}$-modules is a module over the ring of germs of $\mathcal{F}$ in x.

Exercise 6.11. Let $\mathcal{B}$ be a sheaf with all germs equal 0. Prove that $\mathcal{B} = 0$.

Exercise 6.12 (*). Find a sheaf $\mathcal{F}$ on M with all germs non-zero, and $\mathcal{F}(M)$ zero.

Definition 6.8. A sheaf is called **globally generated** if for any $x \in M$, the natural restriction map $\mathcal{F}(M) \longrightarrow \mathcal{F}_x$ from the space of global sections to the space of germs is surjective.

Exercise 6.13 (*). Let $\mathcal{F}$ be a globally generated sheaf on M, and $U \subset M$ an open subset. Prove that the map $\mathcal{F}(M) \longrightarrow \mathcal{F}(U)$ is always surjective, or find a counterexample.

Exercise 6.14 (*). Let M be a smooth, metrizable manifold, and $\mathcal{F}$ be a sheaf of modules over $C^\infty(M)$. Prove that $\mathcal{F}$ is globally generated.

Definition 6.9. A free sheaf of modules $\mathcal{F}^n$ over a ring sheaf $\mathcal{F}$ maps an open set U to the space $\mathcal{F}(U)^n$. A sheaf of $\mathcal{F}$-modules is **non-free** if it is not isomorphic to a free sheaf.

Exercise 6.15 (!). Find a subsheaf of modules in $C^\infty M$ which is non-free in the sense of this definition.

Definition 6.10. **Locally free sheaf of modules** over a sheaf of rings $\mathcal{F}$ is a sheaf of modules $\mathcal{B}$ satisfying the following condition. For each $x \in M$ there exists a neighbourhood $U \ni x$ such that the restriction $\mathcal{B}\big|_U$ is free.

Exercise 6.16. Prove that a sheaf of $C^\infty M$-modules $\operatorname{Der}_{\mathbb{R}}(C^\infty M)$ is locally free, for each manifold M.

Exercise 6.17. Prove that $\operatorname{Der}_{\mathbb{R}}(C^\infty M)$ is a free sheaf for the following manifolds.

a. $M = \mathbb{R}$

b. $M = S^1$ (a circle)

c. $M = \mathbb{R}^2/\mathbb{Z}^2$ (a torus)

d. (*) $M = S^3$ (a three-dimensional sphere)

Exercise 6.18 (*). Find a manifold for which the sheaf $\operatorname{Der}_{\mathbb{R}}(C^\infty M)$ is not free.

Definition 6.11. **A vector bundle** on a ringed space $(M, \mathcal{F})$ is a locally free sheaf of $\mathcal{F}$-modules.

Definition 6.12. The sheaf of C^∞-modules $\operatorname{Der}_{\mathbb{R}}(C^\infty M)$ is called **a tangent bundle** to M.

Exercise 6.19 (!). Let B be a vector bundle on a manifold $(M, C^\infty M)$. Prove that B is globally generated (as a sheaf).

Exercise 6.20 (**). Let B_1, B_2 be vector bundles on (M, C^∞) such that the spaces of sections $B_1(M)$ and $B_2(M)$ are isomorphic as $C^\infty(M)$-modules. Prove that the bundles B_1 and B_2 are isomorphic.

Exercise 6.21 (!). Let $\mathcal{F}$ be a locally free sheaf of $C^\infty M$-modules. Prove that $\mathcal{F}$ is soft.

Exercise 6.22 (**). Let $\mathcal{F}$ be a sheaf of $C^\infty M$-modules. Prove that $\mathcal{F}$ is soft, or find a counterexample.

Definition 6.13. Let $\mathcal{F}$ be a sheaf of $C^\infty M$-modules, and $\mathcal{F}_x$ its germ in x. Denote the quotient $\mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x$ by $\mathcal{F}\big|_x$. This space is called **the fiber** of $\mathcal{F}$ in x. A morphism of sheaves induces a linear map on each of its fibers.

Exercise 6.23 (**). Let $\mathcal{F}$ be a sheaf of $C^\infty M$-modules such that all its fibers $\mathcal{F}\big|_x$ vanish. Prove that $\mathcal{F}$ is zero, or find a counterexample.