Geometry 7: Smooth fibrations

Rules: You may choose to solve only "hard" exercises (marked with !, * and **) or "ordinary" ones (marked with ! or unmarked), or both, if you want to have extra problems. To have a perfect score, a student must obtain (in average) a score of 10 points per week. It's up to you to ignore handouts entirely, because passing tests in class and having good scores at final exams could compensate (at least, partially) for the points obtained by grading handouts.

Solutions for the problems are to be explained to the examiners orally in the class and marked in the score sheet. It's better to have a written version of your solution with you. It's OK to share your solutions with other students, and use books, Google search and Wikipedia, we encourage it.

If you have got credit for 2/3 of ordinary problems or 2/3 of "hard" problems, you receive $6t$ points, where t is a number depending on the date when it is done. Passing all "hard" or all "ordinary" problems (except at most 2) brings you $10t$ points. Solving of "**" (extra hard) problems is not obligatory, but each such problem gives you a credit for 2 "*" or "!" problems in the "hard" set.

The first 3 weeks after giving a handout, $t = 1.5$, between 21 and 35 days, $t = 1$, and afterwards, $t = 0.7$. The scores are not cumulative, only the best score for each handout counts.

Please keep your score sheets until the final evaluation is given.

7.1 Locally trivial smooth fibrations

Definition 7.1. Let $M \xrightarrow{\phi} N$ be a differentiable map of smooth manifolds. A **critical point** of ϕ is a point $m \in M$ where its differential has rank less than maximal possible: $r < \min(\dim M, \dim N)$.

Exercise 7.1. Let $M \xrightarrow{\phi} N$ be a map without critical points, $\dim M > \dim N$, and $X \subset N$ a smooth submanifold. Prove that $\phi^{-1}(X)$ is a smooth submanifold in M.

Hint. Use the implicit function theorem.

Definition 7.2. A trivial smooth fibration is a projection $N \times U \longrightarrow U$, where N and U are smooth manifolds.

Definition 7.3. A surjective smooth map of manifolds $M \xrightarrow{\phi} N$ is called **a locally trivial smooth fibration** if each $x \in N$ has a neighbourhood $U \ni x$ such that the projection $\phi^{-1}(U) \longrightarrow U$ is a trivial smooth fibration.

Remark 7.1. Let $M \xrightarrow{\phi} N$ be a locally trivial smooth fibration, and $U \subset N$ an open subset. The map $\phi^{-1}(U) \longrightarrow U$ is called **restriction of the locally trivial fibration to $U \subset N$.**

Exercise 7.2. Show that any locally trivial fibration is a map without critical points.

Exercise 7.3. Consider a 3-dimensional sphere $S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$, and let $\pi : S^3 \longrightarrow \mathbb{C}P^1$ be a projection induced by the tautological map $\mathbb{C}^2\backslash 0 \longrightarrow \mathbb{C}P^1$. Show that it is a locally trivial fibration with fiber S^1.

Remark 7.2. This map is called **the Hopf fibration**.

Exercise 7.4. Let $\pi : S^3 \longrightarrow \mathbb{C}P^1$ be a Hopf fibration and $\mathbb{C} = \mathbb{C}P^1\backslash\{0\} \hookrightarrow \mathbb{C}P^1$ the standard embedding. Prove that $\pi^{-1}(\mathbb{C}P^1\backslash\{0\})$ is homeomorphic to $S^1 \times \mathbb{R}^2$.

Exercise 7.5. Prove that Hopf fibration is not a trivial fibration.

Exercise 7.6 (**). Prove **Ehresmann theorem**: any surjective, smooth map of compact manifolds without critical points is a locally trivial fibration.

Exercise 7.7. a. Construct a surjective map $S^{2n+1} \longrightarrow \mathbb{C}P^n$ without critical points.

b. (**) Prove that this is a locally trivial, but non-trivial fibration.

Hint. Generalize the construction of Hopf fibration.

Exercise 7.8 (**). Construct a locally trivial smooth fibration $S^7 \longrightarrow S^4$. Prove that it is non-trivial.

Definition 7.4. Let $M_1 \xrightarrow{\pi_1} N$ and $M_2 \xrightarrow{\pi_2} N$ be continuous maps of topological spaces, and $\Delta \subset N \times N$ a diagonal. Let $\pi_1 \times \pi_2 : M_1 \times M_2 \longrightarrow N \times N$ be a natural projection. Define $M_1 \times_N M_2 := (\pi_1 \times \pi_2)^{-1}(\Delta)$. The space $M_1 \times_N M_2$ is called **a fibered product**, or **fiber product of M_1 and M_2 over N.**

Exercise 7.9 (!). Let $M_1 \xrightarrow{\pi_1} N$ and $N_2 \xrightarrow{\pi_2} N$ be locally trivial smooth fibrations with fibers F_1 and F_2. Prove that the natural map $M_1 \times_N M_2 \longrightarrow N$ is a locally trivial fibration with fiber $F_1 \times F_2$.

Exercise 7.10. Represent a Moebius strip as a smooth fibration $\xrightarrow{\pi} S^1$ with fiber $]0,1[$. Prove that $M \times_{S^1} M$ is homeomorphic to $S^1 \times]0,1[\times]0,1[$.

Exercise 7.11 (*). Let $\pi : S^3 \longrightarrow \mathbb{C}P^1$ be a Hopf fibration. Prove that $S^3 \times_{\mathbb{C}P^1} S^3$ is homeomorphic to $S^3 \times S^1$.

7.2 Groups and fiber products

Definition 7.5. A topological group is a topological space equipped with the group operations (product and taking inverse) which are continuous and satisfy the group axioms.

Exercise 7.12. Let G be a subgroup of the group of matrices, with natural topology. Prove that it is a topological group.

Exercise 7.13. Construct a structure of topological group on S^3.

Exercise 7.14 (*). Can an even-dimensional sphere be a topological group?

Exercise 7.15 (*). Can a bouquet of two circles be a topological group?

Definition 7.6. Let $M \xrightarrow{f} N$, $M' \xrightarrow{f'} N$ be continuous maps (morphisms) of topological spaces. A morphism $M \xrightarrow{\psi} M'$ is called **a morphism over** N, if the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{\psi} & M' \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow \\ N & \xrightarrow{\mathsf{Id}} & N \end{array}$$

Definition 7.7. Let $B \xrightarrow{\pi} M$ be a continuous map, and $B \times_M B \xrightarrow{\Psi} M$ - a morphism over M. This morphism is called **associative multiplication** if it is associative on the fibers of π, that is, satisfies $\Psi(a, \Psi(b, c)) = \Psi(\Psi(a, b), c)$ for every triple a, b, c in the same fiber. A section $M \xrightarrow{e} B$ is called **the unit** if the maps

$$B \xrightarrow{\mathsf{Id}_B \times e} B \times_M B \xrightarrow{\Psi} B$$

and

$$B \xrightarrow{e \times \mathsf{Id}_B} B \times_M B \xrightarrow{\Psi} B$$

are equal to Id_B. A morphism $\nu : B \longrightarrow B$ over M is called **group inverse** if each of the maps

$$B \xrightarrow{\Delta} B \times_M B \xrightarrow{\mathsf{Id}_B \times \nu} B \times_M B \xrightarrow{\Psi} B$$

and

$$B \xrightarrow{\Delta} B \times_M B \xrightarrow{\nu \times \mathsf{Id}_B} B \times_M B \xrightarrow{\Psi} B$$

is a constant map, mapping b to $e(\pi(b))$. A map $B \xrightarrow{\pi} M$ equipped with associative multiplication, unit and group inverse is called **a topological group over** M.

Exercise 7.16. Let $B \xrightarrow{\pi} M$ be a topological group over M. Show that all fibers of π ara topological groups.

Exercise 7.17. Let $G \times M \longrightarrow M$ be a trivial fibration. Assume that G is equipped with a set of continuous group operations, indexed by $m \in M$ and continuously depending on m (that is, the corresponding maps, say, $G \times G \times M \longrightarrow G$ are continuous). Prove that this date gives a structure of topological group over M on $G \times M$.

Exercise 7.18. Let B be a topological group over M. Consider the space of continuous sections $M \longrightarrow B$. Prove that it is a group.

7.3 Vector bundles and smooth fibrations

Exercise 7.19. Let G be an abelian group, and k a field. Suppose that for each non-zero $\lambda \in k$ there exists an automorphism $\phi_\lambda : G \longrightarrow G$, such that $\phi_\lambda \circ \phi_{\lambda'} = \phi_{\lambda\lambda'}$, and $\phi_{\lambda+\lambda'}(g) = \phi_\lambda(g) + \phi_{\lambda'}(g)$. Show that G is a vector space over k. Show that all vector spaces can be obtained this way.

Definition 7.8. Let $k = \mathbb{R}$ or $\mathbb{C}$. An abelian topological group $B \xrightarrow{\pi} M$ over M is called **relative vector space over** M if for each non-zero $\lambda \in k$ there exists a continuous automorphism $\phi_\lambda : B \longrightarrow B$ of a group B over M satisfying assumptions of Exercise 7.19, such that the corresponding map $B \times k \longrightarrow B$ is continuous.

Exercise 7.20. Let $B \xrightarrow{\pi} M$ be a relative vector space over M, $U \subset M$ an open subset, and $\mathcal{B}(U)$ the space of sections of a map $\pi^{-1}(U) \xrightarrow{\pi} U$.

a. Show that $\mathcal{B}(U)$ is a vector space.

b. Prove that $\mathcal{B}(U)$ defines a sheaf of modules over a sheaf $C^0(M)$ of continuous functions.

Exercise 7.21. Let $S \subset \mathbb{R}^n$ be a subset (not necessarily a smooth submanifold), $s \in S$ a point, and $v \in T_s\mathbb{R}^n$ a vector. We sat that v belongs to a **tangent cone** C_sS if the distance from S to a point $s + tv$ converges to 0 as $t \to 0$ faster than linearly:

$$\lim_{t \longrightarrow 0} \frac{d(S, s+tv)}{t} \longrightarrow 0.$$

a. (!) Let T_sS be a space generated by C_sS. Show that the set TS of all pairs $(s, v), s \in S, v \in T_sS$ is a relative vector space over S.[1]

b. (!) Find CS for set $S \subset \mathbb{R}^3$ of zeros of a polynomial $x^2 + y^2 - z^2$.

c. (!) Show that in this situation, $CS \longrightarrow S$ is not a locally trivial smooth fibration.

Definition 7.9. Let $B \longrightarrow M$ be a smooth locally trivial fibration with fiber $\mathbb{R}^n$. Assume that B is equipped with a structure of relative vector space over M, and all the maps used in the definition of a relative vector space are smooth. Then B is called **a total space of a vector bundle.**

Exercise 7.22. Let $B \longrightarrow M$ be a relative vector space over M, and $\mathcal{F}$ the corresponding sheaf of sections. Prove that it is a locally free sheaf of $C^\infty M$-modules.

Definition 7.10. Recall that **a vector bundle** is a locally free sheaf of modules over $C^\infty M$. A vector bundle is called **trivial** if it is isomorphic to $C^\infty M^n$.

Definition 7.11. Let $\mathcal{B}$ be an n-dimensional vector bundle on M, $x \in M$ a point, $\mathcal{B}_x$ the space of germs of $\mathcal{B}$ in x, and $\mathfrak{m}_x \subset C^\infty_x M$ the maximal ideal in the ring of germs $C^\infty_x M$ of smooth functions. Define **the fiber** of $\mathcal{B}$ in x as a quotient $\mathcal{B}_x / \mathfrak{m}_x \mathcal{B}_x$. A fiber of a vector bundle is denoted $\mathcal{B}\big|_x$.

Exercise 7.23. Show that a fiber of an n-dimensional bundle is an n-dimensional vector space.

Exercise 7.24. Let $\mathcal{B} = C^\infty M^n$ be a trivial n-dimensional bundle on M, and $b \in \mathcal{B}\big|_x$ a point of a fiber, represented by a germ $\phi \in \mathcal{B}_x = C^\infty_m M^n$, $\phi = (f_1, ..., f_n)$. Consider a map from the set of all fibers $\mathcal{B}$ to $M \times \mathbb{R}^n$, mapping $(x, \phi = (f_1, ..., f_n))$ to $(f_1(x), ..., f_n(x))$. Prove that this map is bijective.

Definition 7.12. Let $\mathcal{B}$ be an n-dimensional vector bundle over M. Denote the set of all vectors in all fibers of $\mathcal{B}$ over all points of M by $\operatorname{Tot}\mathcal{B}$. Let $U \subset M$ be an open subset of M, with $\mathcal{B}\big|_U$ a trivial bundle. Using the local bijection $\operatorname{Tot}\mathcal{B}(U) = U \times \mathbb{R}^n$ defined in Exercise 7.24, we consider topology on $\operatorname{Tot}\mathcal{B}$ induced by open subsets in $\operatorname{Tot}\mathcal{B}(U) = U \times \mathbb{R}^n$ for all open subsets $U \subset M$ and all trivializations of $\mathcal{B}\big|_U$.

[1]The space CS is called **a tangent cone to** S.

Exercise 7.25. Show that $\operatorname{Tot}\mathcal{B}$ with this topology is a locally trivial fibration over M, with fiber $\mathbb{R}^n$.

Exercise 7.26 (!). Show that $\operatorname{Tot}\mathcal{B}$ is equipped with a natural structure of a relative vector space over M, and the sheaf of smooth sections of $\operatorname{Tot}\mathcal{B} \longrightarrow M$ is isomorphic to $\mathcal{B}$.

Definition 7.13. Let $\mathcal{B}$ be a vector bundle on M. Then $B = \operatorname{Tot}\mathcal{B}$ is called **the total space of a vector bundle** $\mathcal{B}$.

Remark 7.3. In practice, "the total space of a vector bundle" is usually denoted by the same letter as the corresponding sheaf. Quite often, mathematicians don't even distinguish between these two notions.

Exercise 7.27. Let $M_1 \xrightarrow{\phi} M$ be a smooth map of manifolds, and $B \xrightarrow{\pi} M$ a total space of a vector bundle. Prove that $B \times_M M_1$ is a total space of a vector bundle on M_1.

Definition 7.14. This bundle is denoted $\phi^* B$, and called **inverse image**, or **a pullback** of B.

Exercise 7.28. Prove that the fiber $\phi^*(B)\big|_x$ is naturally identified with $B\Big|_{\phi(x)}$.

Exercise 7.29. Prove that a pullback of a trivial bundle is trivial.

Exercise 7.30. Let $M_1 \xrightarrow{\phi} M$ be a surjective, smooth map without critical points, and B a non-trivial bundle on M.

a. (*) Can the bundle $\phi^* B$ be trivial?

b. (*) Suppose that M_1 is compact. Can $\phi^* B$ be trivial?