Geometry 8: Vector bundles

Rules: You may choose to solve only "hard" exercises (marked with !, * and **) or "ordinary" ones (marked with ! or unmarked), or both, if you want to have extra problems. To have a perfect score, a student must obtain (in average) a score of 10 points per week. It's up to you to ignore handouts entirely, because passing tests in class and having good scores at final exams could compensate (at least, partially) for the points obtained by grading handouts.

Solutions for the problems are to be explained to the examiners orally in the class and marked in the score sheet. It's better to have a written version of your solution with you. It's OK to share your solutions with other students, and use books, Google search and Wikipedia, we encourage it.

If you have got credit for 2/3 of ordinary problems or 2/3 of "hard" problems, you receive $6t$ points, where t is a number depending on the date when it is done. Passing all "hard" or all "ordinary" problems (except at most 2) brings you $10t$ points. Solving of "**" (extra hard) problems is not obligatory, but each such problem gives you a credit for 2 "*" or "!" problems in the "hard" set.

The first 3 weeks after giving a handout, $t = 1.5$, between 21 and 35 days, $t = 1$, and afterwards, $t = 0.7$. The scores are not cumulative, only the best score for each handout counts.

Please keep your score sheets until the final evaluation is given.

8.1 Tensor product

Definition 8.1. Let V, V' be R-modules, W a free abelian group generated by $v \otimes v'$, with $v \in V, v' \in V'$, and $W_1 \subset W$ a subgroup generated by combinations $rv \otimes v' - v \otimes rv'$, $(v_1 + v_2) \otimes v' - v_1 \otimes v' - v_2 \otimes v'$ and $v \otimes (v'_1 + v'_2) - v \otimes v'_1 - v \otimes v'_2$. Define **the tensor product** $V \otimes_R V'$ as a quotient group W/W_1.

Exercise 8.1. Show that $r \cdot v \otimes v' \mapsto (rv) \otimes v'$ defines an R-module structure on $V \otimes_R V'$.

Exercise 8.2. Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) = 0$.

Exercise 8.3 (*). Find a non-zero R-module V such that $V \otimes_R V = 0$.

Exercise 8.4. Let I_1, I_2 be ideals in R. Prove that $(R/I_1) \otimes_R (R/I_2) = R/(I_1 + I_2)$, where $I_1 + I_2$ is an ideal generated by linear combinations I_1, I_2.

Exercise 8.5. Prove that a tensor product of free R-modules is free.

Exercise 8.6. Let $\mathcal{F}$ be a sheaf of rings, and $\mathcal{B}_1$ and $\mathcal{B}_2$ be sheaves of locally free $(M, \mathcal{F})$-modules. Prove that

$$U \longrightarrow \mathcal{B}_1(U) \otimes_{\mathcal{F}(U)} \mathcal{B}_2(U)$$

is also a sheaf of modules.

Exercise 8.7 (**). Is the last statement true without the assumption of local triviality?

Definition 8.2. Tensor product of vector bundles is a tensor product of the corresponding sheaves of modules.

Remark 8.1. In a similar way one defines exterior powers and symmetric powers of a bundle.

Exercise 8.8. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be locally free sheaves of $C^\infty M$-modules, and $\mathcal{B}_1 \otimes_{C^\infty M} \mathcal{B}_2$ their tensor product. Show that the fiber $\mathcal{B}_1 \otimes_{C^\infty M} \mathcal{B}_2$ in x is naturally identified with a tensor product of the fibers:

$$\Big(\mathcal{B}_1 \otimes_{C^\infty M} \mathcal{B}_2\Big)\Big|_x \cong \mathcal{B}_1\Big|_x \otimes_{\mathbb{R}} \mathcal{B}_2\Big|_x .$$

Exercise 8.9. Let V be an R-module, and $\operatorname{Hom}_R(V, R)$ the space of R-linear homomorphisms from V to R. Prove that the action $r \cdot h(\dots) \mapsto rh(\dots)$ gives a structure of R-module on $\operatorname{Hom}_R(V, R)$. Prove that $\operatorname{Hom}_R(R^n, R)$ with R-module structure defined this way is isomorphic (non-canonically) to a free module R^n.

Definition 8.3. Let V be an R-module. **A dual R-module** V^* is $\operatorname{Hom}_R(V, R)$ with the R-module structure defined above.

Exercise 8.10. Consider $\mathbb{Q}/Z$ as a Z-module. Prove that $(\mathbb{Q}/Z)^* = 0$.

Exercise 8.11. Prove that $\operatorname{Hom}_{\mathbb{Z}}(Q, \mathbb{Z}) = 0$.

Exercise 8.12 (*). Let $R = C^\infty(\mathbb{R})_0$ be a ring of germs of smooth functions at 0, and K an ideal of functions vanishing in 0 with all derivatives. Prove that $(R/K)^* := \operatorname{Hom}_R(R/K, R) = 0$, or disprove it.

Exercise 8.13 (*). Same question when $R = C^\infty(\mathbb{R}^n)_0$.

Exercise 8.14 (!). Let $\mathcal{B}$ be a vector bundle, that is, a locally free sheaf of $C^\infty M$-modules, and $\operatorname{Tot}\mathcal{B} \xrightarrow{\pi} M$ its total space. Define $\mathcal{B}^*(U)$ as a space of smooth functions on $\pi^{-1}(U)$ linear in the fibers of π.

a. Show that the natural restriction map $\mathcal{B}^*(U) \longrightarrow \mathcal{B}^*(V)$ defines a sheaf $\mathcal{B}^*$.

b. Show that this sheaf is locally trivial.

c. (!) Show that $\mathcal{B}^*(U)$ is a dual $C^\infty(U)$-module to $\mathcal{B}(U)$.

Definition 8.4. Let $\mathcal{B}$ be a vector bundle, and $\mathcal{B}^*$ a locally trivial sheaf of $C^\infty M$-modules defined above. It is called **the dual bundle** to $\mathcal{B}$.

Exercise 8.15. Prove that the fiber $\mathcal{B}^*\big|_x$ is a vector space dual to $\mathcal{B}\big|_x$.

Exercise 8.16. Let $\mathcal{B}$ be a non-trivial vector bundle. Prove that $\mathcal{B}^*$ is also non-trivial.

Definition 8.5. Bilinear form on a bundle $\mathcal{B}$ is a section of $(\mathcal{B} \otimes \mathcal{B})^*$. A symmetric bilinear form on $\mathcal{B}$ is called **positive definite** if it gives a positive definite form on all fibers of $\mathcal{B}$. Symmetric positive definite form is also called **a metric**. A skew-symmetric bilinear form on $\mathcal{B}$ is called **non-degenerate** if it is non-degenerate on all fibers of $\mathcal{B}$.

Exercise 8.17 (!). Let $\mathcal{B}$ be a vector bundle on a metrizable manifold M. Prove that $\mathcal{B}$ admits a metric.

Hint. Construct the metric locally, and use partition of unity.

Exercise 8.18. Construct a 2-dimensional vector bundle which does not admit a non-degenerate skew-symmetric bilinear form.

Exercise 8.19 (**). Let M be a simply connected manifold, and B a $2n$-dimensional vector bundle. Prove that B admits a non-degenerate skew-symmetric bilinear form, or find a counterexample.

Exercise 8.20 (*). Find a non-trivial 3-dimensional bundle $\mathcal{B}$ such that its exterior square $\Lambda^2\mathcal{B}$ is trivial.

Exercise 8.21 (*). Find a 2-dimensional bundle which does not admit a non-degenerate bilinear symmetric form of signature $(1,1)$.

8.2 Smooth morphisms of vector bundles and subbundles

Definition 8.6. Let $\mathcal{B}, \mathcal{B}'$ be sheaves on M. **A sheaf morphism** from $\mathcal{B}$ to $\mathcal{B}'$ is a collection of homomorphisms $\mathcal{B}(U) \longrightarrow \mathcal{B}'(U)$, defined for each open subset $U \subset M$, and compatible with the restriction maps:

$$\begin{array}{ccc} \mathcal{B}(U) & \longrightarrow & \mathcal{B}'(U) \\ \downarrow & & \downarrow \\ \mathcal{B}(U_1) & \longrightarrow & \mathcal{B}'(U_1) \end{array}$$

Remark 8.2. Morphisms of sheaves of modules are defined in the same way, but in this case the maps $\mathcal{B}(U) \longrightarrow \mathcal{B}'(U)$ should be compatible with the module structure.

Definition 8.7. A sheaf morphism is called **injective** if it is injective on germs and **surjective**, if it is surjective on germs.

Exercise 8.22. Let $\mathcal{B} \xrightarrow{\phi} \mathcal{B}'$ be an injective morphism of sheaves on M. Prove that ϕ induces an injective map $\mathcal{B}(M) \longrightarrow \mathcal{B}'(M)$ on the spaces of global sections.

Exercise 8.23 (*). Find an example of a surjective sheaf morphism which is not surjective on global sections.

Definition 8.8. Let $\mathcal{B} \xrightarrow{\phi} \mathcal{B}'$ be a morphism of locally free sheaves of $C^\infty M$-modules. It is called **a smooth morphism**, or **a morphism of vector bundles** if on each of the germ spaces ϕ has free kernel and free cokernel.

Definition 8.9. Let $\mathcal{F}$ be a locally free sheaf of $C^\infty M$-modules, and $\mathcal{F}_x$ its space of germs in x. Denote the quotient $\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$ by $\mathcal{F}\big|_x$. This space is called **the fiber** of $\mathcal{F}$ in x. A morphism of sheaves induces a linear map on each of its fibers.

Exercise 8.24 (!). Find an example of an injective morphism of locally free $C^\infty M$-modules which is not injective in some fiber.

Exercise 8.25 (*). Prove that a surjective morphism of locally free sheaves of $C^\infty M$-modules is a smooth morphism of vector bundles, in the sense of the above definition.

Exercise 8.26. Let $\mathcal{B} \longrightarrow \mathcal{B}_1$ be a smooth morphism of vector bundles on M.

a. Prove that the corresponding map Ψ of total spaces is a homomorphism of relative vector spaces over M.

b. Prove that Ψ has no critical points.

Definition 8.10. A subbundle $\mathcal{B}_1 \subset \mathcal{B}$ is an image of an injective morphism of vector bundles.

Exercise 8.27. Let $\mathcal{B}_1 \subset \mathcal{B}$ be a subbundle. Prove that the quotient $\mathcal{B}/\mathcal{B}_1$ is also a vector bundle.

Exercise 8.28 (!). Let $\mathcal{B}_1 \xrightarrow{\phi} \mathcal{B}_2$ be a morphism of vector bundles. Prove that the image of ϕ is a subbundle in $\mathcal{B}_2$, and its kernel is a subbundle in $\mathcal{B}_1$.

Definition 8.11. Direct sum of vector bundles is a direct sum of corresponding sheaves.

Exercise 8.29. Prove that a total space of a direct sum of vector bundles $\mathcal{B}\oplus\mathcal{B}'$ is homeomorphic to $\operatorname{Tot}\mathcal{B} \times_M \operatorname{Tot}\mathcal{B}'$.

Exercise 8.30. Let $\mathcal{B}$ be a vector bundle equipped with a metric (that is, a positive definite symmetric form), and $\mathcal{B}_1 \subset \mathcal{B}$ a subbundle. Consider a subset $\operatorname{Tot}\mathcal{B}_1^\perp \subset \operatorname{Tot}\mathcal{B}$, consisting of all $v \in \mathcal{B}\big|_x$ orthogonal to $\mathcal{B}_1\big|_x \subset \mathcal{B}\big|_x$. Prove that $\operatorname{Tot}\mathcal{B}_1^\perp$ is a total space of a subbundle, denoted as $\mathcal{B}_1^\perp \subset \mathcal{B}$.

Definition 8.12. A subbundle $\mathcal{B}_1^\perp \subset \mathcal{B}$ is called **orthogonal complement** of $\mathcal{B}$ to $\mathcal{B}_1 \subset \mathcal{B}$.

Exercise 8.31. Let $\mathcal{B}_1 \subset \mathcal{B}$ be a sub-bundle. Prove that $\mathcal{B}$ is isomorphic to a direct sum of $\mathcal{B}_1$ and another bundle.

Hint. Find a metric on $\mathcal{B}$ and use the previous exercise.

Remark 8.3. In this situation, it is said that $\mathcal{B}_1$ **is a direct sum of** $\mathcal{B}$.