Geometry 9: Serre-Swan theorem

Rules: You may choose to solve only "hard" exercises (marked with !, * and **) or "ordinary" ones (marked with ! or unmarked), or both, if you want to have extra problems. To have a perfect score, a student must obtain (in average) a score of 10 points per week. It's up to you to ignore handouts entirely, because passing tests in class and having good scores at final exams could compensate (at least, partially) for the points obtained by grading handouts.

Solutions for the problems are to be explained to the examiners orally in the class and marked in the score sheet. It's better to have a written version of your solution with you. It's OK to share your solutions with other students, and use books, Google search and Wikipedia, we encourage it.

If you have got credit for 2/3 of ordinary problems or 2/3 of "hard" problems, you receive $6t$ points, where t is a number depending on the date when it is done. Passing all "hard" or all "ordinary" problems (except at most 2) brings you $10t$ points. Solving of "**" (extra hard) problems is not obligatory, but each such problem gives you a credit for 2 "*" or "!" problems in the "hard" set.

The first 3 weeks after giving a handout, $t = 1.5$, between 21 and 35 days, $t = 1$, and afterwards, $t = 0.7$. The scores are not cumulative, only the best score for each handout counts.

Please keep your score sheets until the final evaluation is given.

9.1 Vector bundles and Whitney theorem

Exercise 9.1. Let $M \subset \mathbb{R}^n$ be a smooth submanifold of $\mathbb{R}^n$, and $TM \subset \mathbb{R}^n \times \mathbb{R}^n$ the set of all pairs $(v, x) \in M \times \mathbb{R}^n$, where $x \in M \times \mathbb{R}^n$ is a point of M, and $v \in \mathbb{R}^n$ a vector tangent to M in m, that is, satisfying

$$\lim_{t \longrightarrow 0} \frac{d(M, m + tv)}{t} \longrightarrow 0.$$

a. Prove that the natural additive operation on $TM \subset M \times \mathbb{R}^n$ (addition of the second argument) defines a structure of a (relative) topological group over M on TM.

b. Prove that a multiplication by real numbers defines on TM a structure of a relative vector space over M.

c. Prove that TM is a total space of a vector bundle.

d. (!) Prove that this vector bundle is isomorphic to a tangent bundle, that is, to the sheaf $\operatorname{Der}_{\mathbb{R}}(C^\infty M)$.

Definition 9.1. The tangent bundle of M, as well as its total space, is denoted by TM.

Exercise 9.2. Let M be a metrizable manifold. Prove that the bundle TM is a direct summand of a trivial bundle.

Hint. Apply Whitney's embedding theorem and use the previous exercise.

Exercise 9.3. Let B be a vector bundle on M, and $\operatorname{Tot} B$ its total space. Consider the tangent bundle $T \operatorname{Tot} B$, and let $M \stackrel{\phi}{\hookrightarrow} \operatorname{Tot} B$ be an embedding corresponding to a zero section. Prove that the pullback $\phi^* T \operatorname{Tot} B$ is isomorphic (as a bundle) to the direct sum $TM \oplus B$.

Exercise 9.4 (!). Prove that any vector bundle on a metrizable, connected manifold is a direct summand of a trivial bundle.

Hint. Use exercises 9.3 and 9.2.

Exercise 9.5. Show that the bundle TS^1 is trivial

Exercise 9.6 (!). Let M be a manifold which is not orientable. Prove that the bundle TM is non-trivial.

Exercise 9.7. Prove that any 1-dimensional bundle on a sphere S^2 is trivial.

Exercise 9.8 (*). Let $TS^2 \oplus \mathbb{R}$ be a direct sum of a tangent bundle TS^2 and a trivial 1-dimensional bundle. Is the bundle $TS^2 \oplus \mathbb{R}$ trivial?

Exercise 9.9. Let G be a topological group, diffeomorphic to a manifold, with all group maps smooth (such a group is called **Lie group**). Prove that the tangent bundle TG is trivial.

Exercise 9.10 (*). Find a non-trivial vector bundle on S^3, or prove that it does not exist.

Definition 9.2. **Rank** of a bundle is the dimension of its fibers.

Definition 9.3. A **line bundle** is a bundle of rank 1.

Exercise 9.11. Let M be a simply connected manifold. Prove that any real line bundle on M is trivial.

Definition 9.4. Let B be a vector bundle of rank n, and $\Lambda^n B$ its top exterior product. This bundle is called **determinant bundle** of B.

Definition 9.5. A real vector bundle is called **orientable** if its determinant bundle is trivial.

Exercise 9.12. a. Prove that a direct sum of orientable vector bundles is orientable.

b. Prove that a tensor product of orientable vector bundles is orientable.

Exercise 9.13 (*). Find a non-trivial, orientable 3-dimensional real vector bundle on a 2-dimensional torus, or prove that it does not exist.

Exercise 9.14 (*). Let B be a real vector bundle on S^n of dimension $\geqslant n+1$. Prove that B is trivial, or find a counterexample.

Exercise 9.15 (**). Consider a bundle $\Lambda^2 S^n$ of 2-forms on an n-dimensional sphere. Find all n for which this bundle is trivial.

9.2 Projective modules

Definition 9.6. Let V be an R-module, and $V' \subset V$ its submodule. Assume that V contains a submodule V'', not intersecting V', such that V' together with V'' generate V. In this case, V' and V'' are called **direct summands** of V, and V – **a direct sum** of V' and V''. This is denoted $V = V' \oplus V''$.

Exercise 9.16. Consider a submodule $n\mathbb{Z} \subset Z$. Can it be realized as a direct summand of $\mathbb{Z}$?

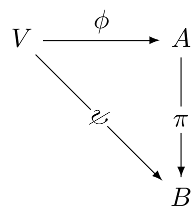
Exercise 9.17. Let R be a ring without zero divisors, and $V = R$ a free module of rank 1. Find all direct summands of V.

Exercise 9.18 (*). Consider a ring of truncated polynomials $R := \mathbb{R}[t]/(t^k)$, and let $V = R$ be a one-dimensional R-module. Find all direct summands of V.

Definition 9.7. An R-module is called **projective** if it is a direct summand of a free module $\bigoplus_I R$ (possibly of infinite rank).

Exercise 9.19. Prove that each R-module is a quotient of a free module.

Exercise 9.20. Let V be an R-module, described below, and $A \xrightarrow{\pi} B$ a surjective homomorphism of R-modules. Prove that each R-module homomorphism $V \xrightarrow{\phi} B$ can be lifted to a morphism $V \xrightarrow{\psi} A$, making the following diagram commutative.



a. Prove it in assumption that V is a free R-module

b. (!) Prove it in assumption that V is projective.

Exercise 9.21 (!). Let V be a module for which the statement of Exercise 9.20 holds true. Prove that V is projective.

Hint. Consider as A a free R-module, mapped to V surjectively, and let B be V, and π an identity map.

Definition 9.8. Let $0 \longrightarrow A \hookrightarrow B \longrightarrow C \longrightarrow 0$ – be an exact sequence of R-modules. Assume that for some $C' \subset B$ one has $B = A \oplus C'$. In this case it is said that the exact sequence $0 \longrightarrow A \hookrightarrow B \longrightarrow C \longrightarrow 0$ **splits**.

Exercise 9.22 (!). Let C be an R-module. Prove that the following conditions are equivalent.

(i) Every exact sequence $0 \longrightarrow A \hookrightarrow B \longrightarrow C \longrightarrow 0$ splits

(ii) The module C is projective.

Exercise 9.23. Let V be a finitely generated projective module over R. Prove that it is free, if

a. (*) $R = \mathbb{Z}$.

b. (*) R is a polynomial ring $\mathbb{C}[t]$.

c. (*) R is a local ring.

Exercise 9.24 (**). A $\mathbb{Z}$-module V is **torsion-free** if the natural map $V \longrightarrow V \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective. Prove that any torsion-free $\mathbb{Z}$-module is projective, or find a counterexample ("finitely generated" is not assumed here).

Exercise 9.25. Let $\mathcal{B}$ be a bundle over a metrizable manifold M, and $\mathcal{B}(M)$ the space of smooth sections of B. Prove that $\mathcal{B}(M)$ is a projective $C^\infty M$-module.

Hint. Use the exercise 9.4.

9.3 Categories and functors

Definition 9.9. **A category** $\mathcal{C}$ is a collection of data ("set of objects of $\mathcal{C}$", "set of morphisms from an object to an object", "operation of composition on morphisms", "identity morphism"), satisfying the following axioms

Objects: The set $\mathcal{O}b(\mathcal{C})$ of objects of $\mathcal{C}$.

Morphisms: For each $X, Y \in \mathcal{O}b(\mathcal{C})$, one is given **the set of morphisms from** X **to** Y, denoted by $\mathcal{M}or(X, Y)$.

Composition of morphisms: If $\phi \in \mathcal{M}or(X, Y), \psi \in \mathcal{M}or(Y, Z)$, one is given the morphism $\phi \circ \psi \in \mathcal{M}or(X, Z)$, called **composition of** ϕ **and** ψ.

Identity morphism: For each $A \in \mathcal{O}b(\mathcal{C})$ one has a distinguished morphism $\mathsf{Id}_A \in \mathcal{M}or(A, A)$.

These data satisfy the following axioms.

Associativity of composition: $\phi_1 \circ (\phi_2 \circ \phi_3) = (\phi_1 \circ \phi_2) \circ \phi_3$.

Properties of identity morphism: For each morphism $\phi \in \mathcal{M}or(X, Y)$, one has $\mathsf{Id}_X \circ \phi = \phi = \phi \circ \mathsf{Id}_Y$.

Exercise 9.26. Prove that the following data define categories.

a. Objects are groups, morphisms are group homomorphisms.

b. Objects are vector spaces, morphisms are linear maps.

c. Objects are vector spaces, morphisms are surjective linear maps.

d. Objects are topological spaces, morphisms – continuous maps.

e. Objects are smooth manifolds, morphisms are smooth maps.

f. Objects – vector bundles on M, morphisms are morphisms of vector bundles.

Definition 9.10. Let $\mathcal{C}_1, \mathcal{C}_2$ be categories. **A covariant functor** from $\mathcal{C}_1$ to $\mathcal{C}_2$ is the following collection of data.

(i) A map $F : \mathcal{O}b(\mathcal{C}_1) \longrightarrow \mathcal{O}b(\mathcal{C}_2)$.

(ii) A map $F : \mathcal{M}or(X, Y) \longrightarrow \mathcal{M}or(F(X), F(Y))$, defined for each $X, Y \in \mathcal{O}b(\mathcal{C}_1)$.

These data define **a functor from** $\mathcal{C}_1$ **to** $\mathcal{C}_2$, if $F(\phi) \circ F(\psi) = F(\phi \circ \psi)$, and $F(\mathsf{Id}_X) = \mathsf{Id}_{F(X)}$.

Exercise 9.27. Let $\mathcal{C}$ be a category of sheaves of modules over a ringed space $(M, \mathcal{F})$. Prove that a correspondence $\mathcal{B} \longrightarrow \mathcal{B}(M)$ defines a functor from $\mathcal{C}$ to a category of $\mathcal{F}(M)$-modules.

Definition 9.11. Two functors $F, G : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ are called **equivalent** if for each $X \in \mathcal{O}b(\mathcal{C}_1)$ there exists an isomorphism $\Psi_X : F(X) \longrightarrow G(X)$, such that for each $\phi \in \mathcal{M}or(X, Y)$ one has

$$F(\phi) \circ \Psi_Y = \Psi_X \circ G(\phi). \tag{9.1}$$

Definition 9.12. A functor $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is called **equivalence of categories** if there exist functors $G, G' : \mathcal{C}_2 \longrightarrow \mathcal{C}_1$ such that $F \circ G$ is equivalent to an identity functor on $\mathcal{C}_1$, and $G' \circ F$ is equivalent to identity functor on $\mathcal{C}_2$.

Exercise 9.28. Prove that the following categories are not equivalent.

a. (!) Category of vector spaces and category of groups.

b. (!) Category of topological spaces and category of vector spaces.

c. (!) Category of groups and category of topological spaces.

Exercise 9.29 (*). Let M be a compact manifold. Prove that the category of sheaves of $C^\infty M$-modules is equivalent to the category of modules over $C^\infty M$, or find a counterexample.

9.4 Serre-Swan theorem

Definition 9.13. Let $x \in M$ be a point on a manifold. **A stalk** of a $C^\infty M$-module V is a tensor product $C^\infty_x M \otimes_{C^\infty M} V$, where $C^\infty_x M$ is a ring of germs of $C^\infty M$ in x. We consider a stalk V_x as a $C^\infty_x M$-module.

Definition 9.14. Recall that **a stalk** of a sheaf F at $x \in M$ is a space of germs of F at x.

Exercise 9.30. Let V be a free $C^\infty M$-module. Prove that a stalk of the space of sections $V(M)$ in x is a stalk of the sheaf V in x.

Definition 9.15. Let $x \in M$ be a point on a manifold. Denote by $\mathfrak{m}_x \subset C^\infty M$ the ideal of all functions vanishing in x. Let $\mathcal{B}$ be a sheaf of $C^\infty M$-modules, and b a section of $\mathcal{B}$. We say that b **nowhere vanishes** if its germ b_x does not lie in $\mathfrak{m}_x\mathcal{B}$ for each $x \in M$.

Exercise 9.31 (!). Let R be a ring of smooth functions on a smooth manifold, $\mathfrak{m}_z$ a maximal ideal of $z \in M$, and B a free R-module, considered as a trivial vector bundle on M of rank n. Let $x_1, ..., x_n \in B$ be a set of sections which are linearly independent in $B/\mathfrak{m}_{z_0}B$ and generate $B/\mathfrak{m}_{z_0}B$, for a fixed point $z_0 \in M$. Let $\xi \in \Lambda^n B$, $\xi := x_1 \wedge x_2 \wedge ... \wedge x_n$ be the determinant of x_i, considered as a section of a line bundle $\det B$. Suppose that ξ nowhere vanishes on $U \subset M$. Prove that $\left\{x_i\big|_U\right\}$ are free generators of $B\big|_U$.

Hint. Define a map $\nu: (C^\infty U)^n \longrightarrow B\big|_U$ mapping generators $e_i \in (C^\infty U)^n$ to x_i. To prove that ν is an isomorphism, use the inverse function theorem.

Exercise 9.32 (!). Let R be a ring of germs of smooth functions on $\mathbb{R}^n$, and V a free R-module, and $V = V_1 \oplus V_2$ a direct sum decomposition. Prove that V_1 and V_2 are also free modules.

Hint. Use the previous exercise.

Exercise 9.33. Let A be a free $C^\infty M$-module, decomposed as a direct sum of two projective modules: $A = B \oplus C$. We identify A with a space of sections of a trivial sheaf of $C^\infty M$-modules, denoted by $\mathcal{A}$. Let $\mathcal{B} \subset \mathcal{A}$ be a subsheaf consisting of all sections $\gamma \in \mathcal{V}(U)$, such that the germs of γ at each $x \in M$ lie in the stalk B_x. Define $\mathcal{C} \subset \mathcal{A}$ in a similar fashion.

a. Prove that $\mathcal{B}$, $\mathcal{C}$ are sheaves of $C^\infty M$-modules.

b. Prove that $\mathcal{A} = \mathcal{B} \oplus \mathcal{C}$.

c. Prove that the stalk B_x of a $C^\infty M$-module B in $x \in M$ is isomorphic to the stalk $\mathcal{B}_x$ of the corresponding sheaf of modules.

Definition 9.16. Let V be a projective $C^\infty M$-module, and $\operatorname{rk}_x V := \dim V/\mathfrak{m}_x V$ dimension of its fiber in x. This number is called **rank** of V in x.

Exercise 9.34. Let V be a projective $C^\infty M$-module, and $x \in M$. Assume that $\operatorname{rk}_x V = n$. Prove that the stalk V_x at x is a free $C^\infty_x M$-module of rank n.

Hint. Use exercise 9.32.

Exercise 9.35. Prove that the rank of a projective $C^\infty M$-module over a connected manifold M is constant.

Hint. Use exercise 9.31.

Definition 9.17. Let B be a projective $C^\infty M$-module, $x_1, ..., x_n$ its sections such that such that their determinant $x_1 \wedge x_2 \wedge ... \wedge x_k \in \Lambda^k \mathcal{B}$ is nowhere vanishing. Then $\{x_i\}$ are called **linearly independent**.

Exercise 9.36. Let $\mathcal{B}$ be a sheaf of $C^\infty M$-modules generated by linearly independent sections $x_1, ..., x_k$. Prove that $\mathcal{B}$ is free.

Exercise 9.37 (!). Let A be a free $C^\infty M$-module, $A = B \oplus C$ its decomposition, and $\mathcal{B}$, $\mathcal{C}$ the corresponding sheaves of modules. Prove each point $x \in M$ has a neighbourhood U such that the sheaf $\mathcal{B}\big|_U$ is generated by $k := \operatorname{rk}_x B$ linearly independent sections $\{x_1, ..., x_k\}$.

Hint. Use exercise 9.31.

Exercise 9.38 (!). Let B be a projective $C^\infty M$-module, and $\mathcal{B}$ a sheaf of modules, generated as in Exercise 9.33. Prove that this sheaf is locally trivial.

Hint. Use the previous exercise.

Exercise 9.39 (!). Let $\mathcal{C}_p$ be a category with objects projective $C^\infty M$-modules, and morphisms homomorphism of $C^\infty M$-modules with kernels and cokernels projective, . Check that the axioms of category are satisfied.

Remark 9.1. Recall that we defined morphisms of vector bundles as morphisms of the corresponding sheaves of $C^\infty M$-modules such that their kernels and cokernels are locally free $C^\infty M$-modules.

Exercise 9.40. (Serre-Swan theorem) Let $\mathcal{C}_b$ be a category of vector bundles on M.

a. (*) Consider a map Ψ making a vector bundle from a projective $C^\infty M$-module, as in Exercise 9.38. Prove that $\Psi(B)$ does not depend on a choice of a free module $A \supset B$.

b. (*) Prove that Ψ defines a functor from $\mathcal{C}_p$ to $\mathcal{C}_b$.

c. (*) Show that this functor defines an equivalence of categories.