Geometry of manifolds

lecture 1

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The Plan.

Preliminaries: I assume knowledge of **topological spaces** and **continuous maps**, **homeomorphisms**, **connected spaces**, **path connected spaces**, **metric spaces**, **compact spaces**, **groups**, **abelian groups**, **homomorphisms** and **vector spaces**.

Plan of today's talk:

1. Topological manifolds.

2. Smooth manifolds.

3. Sheaves of functions.

4. 3 different definitions of a smooth manifold

5. Partition of unity.

Topological manifolds

REMARK: Manifolds can be smooth (of a given "differentiability class"), real analytic, or topological (continuous).

DEFINITION: Topological manifold is a topological space which is locally homeomorphic to an open ball in $\mathbb{R}^n$.

PROBLEM: Show that a group of homeomorphisms acts on a connected manifold transitively.

DEFINITION: Such a topological space is called **homogeneous**.

Topological manifolds: some unsolved problems

DEFINITION: Geodesic in a metric space is an isometry $[0,1] \longrightarrow M$.

DEFINITION: A **Busemann space** is a metric space M such that any two points can be connected by a geodesic, any closed, bounded subset of M is compact, and a geodesic connecting x to y is unique when $d(x,y)$ is sufficiently small.

REMARK: A Busemann space is homogeneous.

CONJECTURE: (Busemann, 1955)
Any Busemann space is a topological manifold.

...Although this (the Busemann Conjecture) is probably true for any G-space, the proof, if the conjecture is correct, seems quite inaccessible in the present state of topology... (Herbert Busemann)

There are many other conjectures about path connected, homogeneous topological spaces (Bing-Borsuk, Moore, de Groot...), implying that they are manifolds, none of them proven, except in low dimension.

Conflict

Herbert Busemann
(Berlin, 1905 - Santa Ynez, 1994)

Conflict, 1972

Atlases on manifolds

DEFINITION: **An open cover** of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$. A cover $\{V_i\}$ is a **refinement** of a cover $\{U_i\}$ if every V_i is contained in some U_i.

REMARK: Any two covers $\{U_i\}$, $\{V_i\}$ of a topological space admit a common refinement $\{U_i \cap V_j\}$.

DEFINITION: Let M be a topological manifold. A cover $\{U_i\}$ of M is an **atlas** if for every U_i, we have a map $\varphi_i : U_i \to \mathbb{R}^n$ giving a homeomorphism of U_i with an open subset in $\mathbb{R}^n$. In this case, one defines the **transition maps**

$$\Phi_{ij} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

DEFINITION: A function $\mathbb{R} \longrightarrow \mathbb{R}$ is **of differentiability class C^i** if it is i times differentiable, and its i-th derivative is continuous. A map $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ is **of differentiability class C^i** if all its coordinate components are. A **smooth function** (map) is a function (map) of class $C^\infty = \bigcap C^i$.

DEFINITION: An atlas is **smooth** if all transition maps are smooth (of class C^∞, i.e., infinitely differentiable), **smooth of class C^i** if all transition functions are of differentiability class C^i, and **real analytic** if all transition maps admit a Taylor expansion at each point.

Smooth structures

DEFINITION: A **refinement** of an **atlas** is a refinement of the corresponding cover $V_i \subset U_i$ equipped with the maps $\varphi_i : V_i \to \mathbb{R}^n$ that are the restrictions of $\varphi_i : U_i \to \mathbb{R}^n$. Two atlases (U_i, φ_i) and (U_i, ψ_i) of class C^∞ or C^i (with the same cover) are **equivalent** in this class if, for all i, the map $\psi_i \circ \varphi_i^{-1}$ defined on the corresponding open subset in $\mathbb{R}^n$ belongs to the mentioned class. Two arbitrary atlases are **equivalent** if the corresponding covers possess a common refinement giving equivalent atlases.

DEFINITION: A **smooth structure** on a manifold (of class C^∞ or C^i) is an atlas of class C^∞ or C^i considered up to the above equivalence. A **smooth manifold** is a topological manifold equipped with a smooth structure.

DEFINITION: A **smooth function** on a manifold M is a function f whose restriction to the chart (U_i, φ_i) gives a smooth function $f \circ \varphi_i^{-1}: \ \varphi_i(U_i) \longrightarrow \mathbb{R}$ for each open subset $\varphi_i(U_i) \subset \mathbb{R}^n$.

Smooth maps and isomorphisms

From now on, **I shall identify the charts U_i with the corresponding subsets of $\mathbb{R}^n$,** and forget the differentiability class.

DEFINITION: A smooth map of $U \subset \mathbb{R}^n$ to a manifold N is a map $f: \ U \longrightarrow N$ such that for each chart $U_i \subset N$, the restriction $f\big|_{f^{-1}(U_i)}$: $f^{-1}(U_i) \longrightarrow U_i$ is smooth with respect to coordinates on U_i. **A map of manifolds** $f: \ M \longrightarrow N$ is **smooth** if for any chart V_i on M, the restriction $f\big|_{V_i}: \ V_i \longrightarrow N$ is smooth as a map of $V_i \subset \mathbb{R}^n$ to N.

DEFINITION: An isomorphism of smooth manifolds is a bijective smooth map $f: \ M \longrightarrow N$ such that f^{-1} is also smooth.

Smooth structures, smooth finctions and sheaves

REMARK: For two equivalent atlases of a given differentiability class C^i, **the spaces $C^i M$ of C^i-functions coincide.**

Converse is also true.

PROBLEM: Let $f: M \longrightarrow N$ be a map of smooth manifolds such that $f^*\mu$ is smooth for any smooth function $\mu: N \longrightarrow \mathbb{R}$. **Show that f is a smooth map.**

REMARK: It's better to define smooth structures in terms of smooth functions, but for practical work **it's most convenient to use sheaves**.

Sheaves

DEFINITION: A **presheaf of functions** on a topological space M is a collection of subrings $\mathcal{F}(U) \subset C(U)$ in the ring $C(U)$ of all functions on U, for each open subset $U \subset M$, such that the restriction of every $\gamma \in \mathcal{F}(U)$ to an open subset $U_1 \subset U$ belongs to $\mathcal{F}(U_1)$.

DEFINITION: A presheaf of functions $\mathcal{F}$ is called **a sheaf of functions** if these subrings satisfy the following condition. Let $\{U_i\}$ be a cover of an open subset $U \subset M$ (possibly infinite) and $f_i \in \mathcal{F}(U_i)$ a family of functions defined on the open sets of the cover and compatible on the pairwise intersections:

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

Sheaves and exact sequences

REMARK: A presheaf of functions is a collection of subrings of functions on open subsets, compatible with restrictions. **A sheaf of fuctions is a presheaf allowing "gluing"** a function on a bigger open set if its restrictions to smaller open sets are compatible.

DEFINITION: A sequence $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow ...$ of homomorphisms of abelian groups or vector spaces is called **exact** if the image of each map is the kernel of the next one.

CLAIM: A presheaf $\mathcal{F}$ is a sheaf if and only if for every cover $\{U_i\}$ of an open subset $U \subset M$, **the sequence of restriction maps**

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$$

is exact, with $\eta \in \mathcal{F}(U_i)$ mapped to $\eta\Big|_{U_i \cap U_j}$ and $-\eta\Big|_{U_j \cap U_i}$.

Sheaves and presheaves: examples

Examples of sheaves:

* Space of continuous functions

* Space of smooth functions, any differentiability class

* Space of real analytic functions

Examples of presheaves which are not sheaves:

* Space of constant functions **(why?)**

* Space of bounded functions **(why?)**

Ringed spaces

A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of functions. A **morphism** $(M, \mathcal{F}) \xrightarrow{\Psi} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \xrightarrow{\Psi} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $\psi^* f := f \circ \Psi$ belongs to the ring $\mathcal{F}\Big(\Psi^{-1}(U)\Big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

EXAMPLE: Let M be a manifold of class C^i and let $C^i(U)$ be the space of functions of this class. **Then C^i is a sheaf of functions, and (M, C^i) is a ringed space.**

REMARK: Let $f: X \longrightarrow Y$ be a smooth map of smooth manifolds. Since a pullback $f^*\mu$ of a smooth function $\mu \in C^\infty(M)$ is smooth, **a smooth map of smooth manifolds defines a morphism of ringed spaces.**

Converse is also true:

Ringed spaces and smooth maps

CLAIM: Let (M, C^i) and (N, C^i) be manifolds of class C^i. Then **there is a bijection between smooth maps $f: M \longrightarrow N$ and the morphisms of corresponding ringed spaces.**

Proof: Any smooth map induces a morphism of ringed spaces. Indeed, **a composition of smooth functions is smooth, hence a pullback is also smooth.**

Conversely, let $U_i \longrightarrow V_i$ be a restriction of f to some charts; to show that f is smooth, it would suffice to show that $U_i \longrightarrow V_i$ is smooth. However, we know that a pullback of any smooth function is smooth. **Therefore, Claim is implied by the following lemma.**

LEMMA: Let M, N be open subsets in $\mathbb{R}^n$ and let $f: M \to N$ map such that a pullback of any function of class C^i belongs to C^i. **Then f is of class C^i.**

Proof: Apply f to coordinate functions. ■

A new definition of a manifold

As we have just shown, this definition is equivalent to the previous one.

DEFINITION: Let $(M, \mathcal{F})$ be a topological manifold equipped with a sheaf of functions. It is said to be a **smooth manifold of class** C^∞ or C^i if every point in $(M, \mathcal{F})$ has an open neighborhood isomorphic to the ringed space $(\mathbb{R}^n, \mathcal{F}')$, where $\mathcal{F}'$ is a ring of functions on $\mathbb{R}^n$ of this class.

DEFINITION: A chart, or **a coordinate system** on an open subset U of a manifold $(M, \mathcal{F})$ is an isomorphism between $(U, \mathcal{F})$ and an open subset in $(\mathbb{R}^n, \mathcal{F}')$, where $\mathcal{F}'$ are functions of the same class on $\mathbb{R}^n$.

DEFINITION: Diffeomorphism of smooth manifolds is a homeomorphism φ which induces an isomorphims of ringed spaces, that is, φ and φ^{-1} map (locally defined) smooth functions to smooth functions.

Assume from now on that all manifolds are Hausdorff and of class C^∞.

Embedded submanifolds

DEFINITION: A **closed embedding** $\varphi: N \hookrightarrow M$ of topological spaces is an injective map from N to a closed subset $\varphi(N)$ inducing a homeomorphism of N and $\varphi(N)$.

DEFINITION: $M \subset \mathbb{R}^n$ is called **a submanifold** of dimension m if for every point $x \in N$, there is a neighborhood $U \subset \mathbb{R}^n$ diffeomorphic to an open ball, such that this diffeomorphism maps $U \cap N$ onto a linear subspace of dimension m.

DEFINITION: A morphism of embedded submanifolds $M_1 \subset \mathbb{R}^n$ to $M_2 \subset \mathbb{R}^n$ is a map $f: M_1 \longrightarrow M_2$ such that any point $x \in M_1$ has a neighbourhood U such that $f\big|_{M_1 \cap U}$ can be extended to a smooth map $U \longrightarrow \mathbb{R}^n$.

REMARK: The third definition of a smooth manifold: **a smooth manifold can be defined as a smooth submanifold in $\mathbb{R}^n$.**

This definition becomes equivalent to the usual one if one proves the Whitney's theorem.

THEOREM: Any manifold can be embedded to $\mathbb{R}^n$.

Its proof takes some work.

Locally finite covers

DEFINITION: An open cover $\{U_\alpha\}$ of a topological space M is called **locally finite** if every point in M possesses a neighborhood that intersects only a finite number of U_α.

CLAIM: Let $\{U_\alpha\}$ be a locally finite atlas on a manifold M. **Then there exists a refinement $\{V_\beta\}$ of $\{U_\alpha\}$ such that a closure of each V_β is compact in M.**

Proof: Let $\{U_\alpha\}$ be a locally finite atlas on M, and $U_\alpha \xrightarrow{\varphi_\alpha} \mathbb{R}^n$ homeomorphisms. Consider a cover $\{V_i\}$ of $\mathbb{R}^n$ given by open balls of radius 2 centered in integer points, and let $\{W_\beta\}$ be a cover of M obtained as union of $\varphi_\alpha^{-1}(V_i)$. **Then $\{W_\beta\}$ is locally finite.** ■

DEFINITION: Let $U \subset V$ be two open subsets of M such that the closure of U is contained in V. **In this case we write $U \Subset V$.**

Locally finite covers and their subcovers

Exercise: Let $U \subset M$ be an open subset with compact closure, and $V \supset M \backslash U$ another open subset. **Prove that there exists $U' \subset U$ such that the closure of U' is contained in U, and $V \cup U' = M$.**

THEOREM: Let $\{U_\alpha\}$ be a countable locally finite cover of a Hausdorff topological space, such that a closure of each U_α is compact. **Then there exists another cover $\{V_\alpha\}$ indexed by the same set, such that $V_\alpha \Subset U_\alpha$.**

Proof. Step 1: Let $U_1, U_2, ...$ be all elements of the cover. Suppose that $V_1, ..., V_{n-1}$ is already found. To take an induction step it remains to find $V_n \Subset U_n$

Step 2: Replacing U_i by V_i and renumbering, we may assume that $n = 1$. **Then the statement of Theorem follows from the previous exercise applied to $V = \bigcup_{i=2}^{\infty} U_i$ and $U = U_1$.** ■

Construction of a partition of unity

REMARK: If all U_α are diffeomorphic to $\mathbb{R}^n$, all V_α can be chosen diffeomorphic to an open ball. Indeed, any compact set is contained in an open ball.

COROLLARY: Let M be a manifold admitting a locally finite countable cover $\{U_\alpha\}$, with $\varphi_\alpha : U_\alpha \longrightarrow \mathbb{R}^n$ diffeomorphisms. **Then there exists another atlas $\{U_\alpha, \varphi'_\alpha : U_\alpha \longrightarrow \mathbb{R}^n\}$, such that $\varphi'_\alpha(\mathbb{B})$ is also a cover of M, and $\mathbb{B} \subset \mathbb{R}^n$ a unit ball.** ■

EXERCISE: Find a smooth function $\nu : \mathbb{R}^n \longrightarrow [0,1]$ which vanishes outside of $\mathbb{B} \subset \mathbb{R}^n$ and is positive on $\mathbb{B}$.

REMARK: In assumptions of Corollary, let $\nu_\alpha(z) := \nu(\varphi'_\alpha)$, and $\mu_i := \frac{\nu_i}{\sum_\alpha \nu_\alpha}$. Then $\mu_\alpha : M \longrightarrow [0,1]$ are smooth functions with support in U_α satisfying $\sum_\alpha \mu_\alpha = 1$. Such a set of functions is called **a partition of unity**.

Partition of unity: a formal definition

DEFINITION: Let M be a smooth manifold and let $\{U_\alpha\}$ a locally finite cover of M. A **partition of unity** subordinate to the cover $\{U_\alpha\}$ is a family of smooth functions $f_i : M \to [0, 1]$ with compact support indexed by the same indices as the U_i's and satisfying the following conditions.
(a) Every function f_i vanishes outside U_i
(b) $\sum_i f_i = 1$

The argument of previous page proves the following theorem.

THEOREM: Let $\{U_\alpha\}$ be a countable, locally finite cover of a manifold M, with all U_α diffeomorphic to $\mathbb{R}^n$. **Then there exists a partition of unity subordinate to $\{U_\alpha\}$.**

Whitney's theorem for compact manifolds

THEOREM: Let M be a compact smooth manifold. **Then M admits a closed smooth embedding to $\mathbb{R}^N$.**

Proof. Step 1: Choose a finite atlas **(why it exists?)** $\{V_i, \varphi_i : V_i \longrightarrow \mathbb{R}^n, i = 1, 2, ..., m\}$, and subordinate partution of unity $\nu_i : M \longrightarrow [0,1]$. For each i, the map

$$\Phi_i(z) := \left(\nu_i \varphi_i(z), \sqrt{1 - \nu_i(z)^2}\right)$$

is injective on the set $\{z \mid \nu_i(z) > 0\}$, and maps M to a sphere $S^n \subset \mathbb{R}^{n+1}$.

Step 2: The product map

$$\prod_{i=1}^m : \Phi_i : M \longrightarrow \underbrace{S^n \times S^n \times ... \times S^n}_{m \text{ times}}$$

is an injective, continuous map from a compact, hence it is a homeomorphism to its image **(prove it)**.

Step 3: Any smooth function on $W_i := \{z \mid \nu_i(z) > \frac{1}{2m}\}$ can be obtained as a restriction of a smooth function on $\mathbb{R}^{n+1}$, hence **this map induces an isomorphism of the corresponding sheaves of smooth functions.** ■