Geometry of manifolds

Lecture 10: de Rham algebra

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Kähler differentials

**DEFINITION:** Let $R$ be a ring over a field $k$, and $V$ an $R$-module. A $k$-linear map $D : R \to V$ is called a derivation if it satisfies the Leibnitz identity $D(ab) = aD(b) + bD(a)$. The space of derivations from $R$ to $V$ is denoted $\text{Der}_k(R,V)$.

**REMARK:** $\text{Der}_k(R,V)$ is an $R$-module, with a natural $R$-action.

**DEFINITION:** Let $R$ be a ring over a field $k$. Define an $R$-module $\Omega^1_k R$ (the module of Kähler differentials) with the following generators and relations.

* Generators of $\Omega^1_k R$ are indexed by elements of $R$; for each $a \in R$, the corresponding generator of $\Omega^1_k R$ is denoted $da$.

* Relations in $\Omega^1_k R$ are generated by expressions $d(ab) = adb + bda$, for all $a, b \in R$, and $d\lambda = 0$ for each $\lambda \in k$.

**EXERCISE:** Prove that the map $d : R \to \Omega^1_k R$ mapping $a$ to $da$ is a derivation.
Universal property of Kähler differentials

**CLAIM:** Let $V$ be an $R$-module, and $D \in \operatorname{Der}_k(R,V)$ a derivation. Then there exists a unique $R$-module homomorphism $\varphi_D : \Omega^1_k R \to V$ mapping $bda$ to $bD(a)$.

**REMARK:** Consider a category $\mathcal{C}$ of $R$-modules equipped with a derivation $(V, D : R \to V)$, and define morphisms in $\mathcal{C}$ as morphisms of $R$-modules which commute with the derivation map. Then $\Omega^1 R$ is an initial object in this category. This is called the universal property of the module of Kähler differentials.

**CLAIM:** $\operatorname{Der}_k(R,V) = \operatorname{Hom}_R(\Omega^1 R, V)$.

**Proof:** A composition of a derivation $R \xrightarrow{d} \Omega^1 R$ and an $R$-module homomorphism $\Omega^1 R \to V$ lies in $\operatorname{Der}_k(R,V)$. On the other hand, any derivation $\xi \in \operatorname{Der}_k(R,V)$ is obtained this way, by the universal property. ■

**COROLLARY:** $\operatorname{Der}_k(R) = (\Omega^1 R)^*$, where $V^* := \operatorname{Hom}(V,R)$. 
Kähler differentials over polynomials

**REMARK:** Unlike the derivations, the Kähler differentials are functorial on $R$.

**CLAIM:** Let $R \xrightarrow{\varphi} R'$ be a ring homomorphism. Consider $\Omega^1 R'$ as an $R$-module, using the action $r, a \mapsto \varphi(r)a$. Then there exists an $R$-module homomorphism $\Omega^1 R \rightarrow \Omega^1 R'$, mapping $dr$ to $d\varphi(r)$.

**CLAIM:** Let $R = k[t_1, \ldots, t_n]$ be a polynomial ring over a field of characteristic 0. Then $\Omega^1_k R$ is a free $R$-module generated by $dt_1, dt_2, \ldots, dt_n$.

**Proof.** *Step 1:* For each polynomial $P \in R$, the element $dP$ can be expressed as a sum $\sum \frac{dP}{dt_i} dt_i$. Therefore, each $\alpha \in \Omega^1 R$ can be written as $\sum_i Q_i dt_i$.

*Step 2:* This expression $\alpha = \sum_i Q_i dt_i$ is unique, because the pairing $\text{Der}_k(R) \times \Omega^1 R \rightarrow R$ maps $\frac{d}{dt_k} \times \sum_i Q_i dt_i$ to $Q_k$, hence the coefficients $Q_k \in R$ are determined unambiguously. ■
Cotangent bundle

**DEFINITION:** Let $A, B$ be finitely generated $R$-modules, and $\nu : A \times B \to R$ a bilinear pairing. Define the **annihilator of $\nu$ in $B$** as a submodule consisting of all elements $b \in B$ for which the homomorphism $\nu(\cdot, b) : A \to R$ vanishes.

**DEFINITION:** Let $M$ be a smooth manifold, $R := C^\infty M$ the ring of smooth functions, and $\nu : \text{Der}(R) \times \Omega^1 R \to R$ the pairing obtained from an isomorphism $\text{Der}(R) = (\Omega^1 R)^*$. Consider its annihilator $K \subset \Omega^1 R$. Define the **cotangent bundle** as $\Lambda^1 M := \Omega^1 R/K$.

**CLAIM:** $\Lambda^1 M$ is generated as a $C^\infty M$-module by $d(C^\infty M)$.

**CLAIM:** In these assumptions, $\Lambda^1 M$ is an image of the tautological map $\Omega^1 M \xrightarrow{\tau} (\Omega^1 M)^{**}$. Moreover, $\Lambda^1(M) = \text{Der}(R)^*$.

**Proof. Step 1:** By construction, $\Lambda^1 M$ is a quotient of $\Omega^1 M$ by a kernel of a map $\Omega^1 M \xrightarrow{\tau} \text{Der}(R)^* = (\Omega^1 M)^{**}$. Therefore, $\Lambda^1 M = \text{im} \tau$.

**Step 2:** Let $V \subset \text{Der}(R)^*$ be a $C^\infty M$-submodule generated by the symbols $df \in \text{Der}(R)^*$ which are paired with vector fields $v$ as $\langle v, df \rangle$. Clearly, $V = \text{im} \tau$. If $M$ admits coordinates $t_1, \ldots, t_n$, one has $V = \text{Der}(R)^*$, because $\text{Der}(R)^*$ is freely generated by $\frac{d}{dt_i}$. 

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Cotangent bundle (cont.)

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Step 2: If $M$ admits coordinates $t_1, \ldots, t_n$, one has $V = \text{Der}(R)^*$, because $\text{Der}(R)^*$ is freely generated by $\frac{d}{dt_i}$.

Step 3: Using partition of unity, we obtain that $V$ is a subsheaf of $\text{Der}(R)^* = (TM)^*$. Indeed, suppose that for some covering $\{U_i\}$ one has $\alpha|_{U_i} = \sum f_j(i)dg_j(i)$, $\varphi_i$ a partition of unity subordinate to $\{U_i\}$, and $\psi_i$ functions with support in $U_i$ satisfying $\psi_i|_{\sup \varphi_i} = 1$. Then $\alpha = \sum \varphi_i f_j(i)d(\psi_i g_j(i))$.

Step 4: Locally in $M$, $V = \text{Der}(R)^*$, because locally $M$ admits coordinates. Therefore, the sheaves $V$ and $\text{Der}(R)^*$ coincide. ■

COROLLARY: $\Lambda^1 M$ is a locally free sheaf of $C^\infty M$-modules. Moreover, $\Lambda^1 M = TM^*$.

COROLLARY: $\Lambda^1 M$ is generated as a $C^\infty M$-module by $d(C^\infty M)$. 6
De Rham algebra

**DEFINITION:** Let $M$ be a smooth manifold. **A bundle of differential $i$-forms on $M$** is the bundle $\Lambda^i T^*M$ of antisymmetric $i$-forms on $TM$. It is denoted $\Lambda^i M$.

**REMARK:** $\Lambda^0 M = C^\infty M$.

**DEFINITION:** Let $\alpha \in (V^*)^i$ and $\alpha \in (V^*)^j$ be polylinear forms on $V$. Define the **tensor multiplication** $\alpha \otimes \beta$ as

$$\alpha \otimes \beta(x_1, ..., x_{i+j}) := \alpha(x_1, ..., x_j)\beta(x_{i+1}, ..., x_{i+j}).$$

**DEFINITION:** Let $\otimes_k T^*M \xrightarrow{\Pi} \Lambda^k M$ be the antisymmetrization map,

$$\Pi(\alpha)(x_1, ..., x_n) := \frac{1}{n!} \sum_{\sigma \in \text{Sym}_n} (-1)^\sigma \alpha(x_{\sigma_1}, x_{\sigma_2}, ..., x_{\sigma_n}).$$

Define **the exterior multiplication** $\wedge : \Lambda^i M \times \Lambda^j M \to \Lambda^{i+j} M$ as $\alpha \wedge \beta := \Pi(\alpha \otimes \beta)$, where $\alpha \otimes \beta$ is a section $\Lambda^i M \otimes \Lambda^j M \subset \otimes^i_j T^* M$ obtained as their tensor multiplication.

**REMARK:** The fiber of the bundle $\Lambda^* M$ at $x \in M$ is **identified with the Grassmann algebra $\Lambda^* T^*_x M$**. This identification is compatible with the Grassmann product.
Coordinate monomials

**DEFINITION:** Let $t_1, ..., t_n$ be coordinate functions on $\mathbb{R}^n$, and $\alpha \in \Lambda^* \mathbb{R}^n$ a monomial obtained as a product of several $dt_i$: $\alpha = dt_{i_1} \wedge dt_{i_2} \wedge ... \wedge dt_{i_k}$, $i_1 < i_2 < ... < i_k$. Then $\alpha$ is called a coordinate monomial.

**CLAIM:** $\Lambda^* \mathbb{R}^n$ is a trivial bundle, and coordinate monomials are free generators of $\Lambda^* \mathbb{R}^n$.

**DEFINITION:** An associative algebra $A^* = \oplus_{i \in \mathbb{Z}} A^i$ is called a graded algebra if for all $a \in A^i$, $b \in A^j$, the product $ab$ lies in $A^{i+j}$.

**EXAMPLE:** De Rham algebra is a graded algebra.
De Rham differential

**DEFINITION: De Rham differential** $d : \Lambda^* M \longrightarrow \Lambda^{*+1} M$ is an $\mathbb{R}$-linear map satisfying the following conditions.

* For each $f \in \Lambda^0 M = \mathcal{C}^\infty M$, $d(f) \in \Lambda^1 M$ is equal to the image of the Kähler differential $df \in \Omega^1 M$ in $\Lambda^1 M = \Omega^1 M / K$.

* (Leibnitz rule) $d(a \wedge b) = da \wedge b + (-1)^j a \wedge db$ for any $a \in \Lambda^i M, b \in \Lambda^j M$.

* $d^2 = 0$.

**REMARK:** A map on a graded algebra which satisfies the Leibnitz rule above is called an **odd derivation**.

**REMARK:** The following two lemmas are needed to prove uniqueness of de Rham differential.

**LEMMA:** Let $A = \bigoplus A^i$ be a graded algebra, $B \subset A$ a set of multiplicative generators, and $D_1, D_2 : A \longrightarrow A$ two odd derivations which are equal on $B$. Then $D_1 = D_2$. ■

**LEMMA:** $\Lambda^* M$ is generated by $\mathcal{C}^\infty M$ and $d(\mathcal{C}^\infty M)$.

**Proof:** By definition, $\Lambda^* M$ is generated by $\Lambda^0 M = \mathcal{C}^\infty M$ and $\Lambda^1 M$. However, $d(\mathcal{C}^\infty M)$ generate $\Lambda^1 M$, as shown above. ■
De Rham differential: uniqueness and existence

**THEOREM:**
De Rham differential is uniquely determined by these axioms.

**Proof:** De Rham differential is an odd derivation. Its value on $C^\infty M$ is defined by the first axiom. On $d(C^\infty M)$ de Rham differential vanishes, because $d^2 = 0$. ■

**DEFINITION:** Let $t_1, ..., t_n$ be coordinate functions on $\mathbb{R}^n$, $\alpha_i$ coordinate monomials, and $\alpha := \sum f_i \alpha_i$. Define $d(\alpha) := \sum_i \sum_j \frac{df_i}{dt_j} dt_j \wedge \alpha_i$.

**EXERCISE:**
Check that $d$ satisfies the properties of de Rham differential.

**COROLLARY:** De Rham differential exists on any smooth manifold.

**Proof:** Locally, de Rham differential $d$ exists, as follows from the construction above. Since $d$ is unique, it is compatible with restrictions. This means that $d$ defines a sheaf morphism. Restricting this sheaf morphism to global sections, we obtain de Rham differential on $\Lambda^* M$. ■
Superalgebras

**DEFINITION:** Let $A^* = \bigoplus_{i \in \mathbb{Z}} A^i$ be a graded algebra over a field. It is called **graded commutative**, or **supercommutative**, if $ab = (-1)^{ij}ba$ for all $a \in A^i, b \in A^j$.

**EXAMPLE:** Grassmann algebra $\Lambda^*V$ is clearly supercommutative.

**DEFINITION:** Let $A^*$ be a graded commutative algebra, and $D : A^* \rightarrow A^{*+i}$ be a map which shifts grading by $i$. It is called a **graded derivation** if $D(ab) = D(a)b + (-1)^{ij}aD(b)$, for each $a \in A^j$.

**REMARK:** If $i$ is even, graded derivation is a usual derivation. If it is even, it an odd derivation.

**DEFINITION:** Let $M$ be a smooth manifold, and $X \in TM$ a vector field. Consider an operation of **convolution with a vector field** $i_X : \Lambda^iM \rightarrow \Lambda^{i-1}M$, mapping an $i$-form $\alpha$ to an $(i-1)$-form $v_1, ..., v_{i-1} \mapsto \alpha(X, v_1, ..., v_{i-1})$

**EXERCISE:** Prove that $i_X$ is an odd derivation.
Supercommutator

**DEFINITION:** Let $A^*$ be a graded vector space, and $E : A^* \to A^{*+i}$, $F : A^* \to A^{*+j}$ operators shifting the grading by $i, j$. Define the supercommutator $\{E, F\} := EF - (-1)^{ij} FE$.

**DEFINITION:** An endomorphism which shifts a grading by $i$ is called **even** if $i$ is even, and **odd** otherwise.

**EXERCISE:** Prove that a supercommutator satisfies **graded Jacobi identity**, 

$$\{E, \{F, G\}\} = \{\{E, F\}, G\} + (-1)^{\tilde{E}\tilde{F}} \{F, \{E, G\}\}$$

where $\tilde{E}$ and $\tilde{F}$ are 0 if $E, F$ are even, and 1 otherwise.

**REMARK:** There is a simple mnemonic rule which allows one to remember a superidentity, if you know the commutative analogue. Each time when in commutative case two letters $A, F$ are exchanged, in supercommutative case one needs to multiply by $(-1)^{\tilde{E}\tilde{F}}$.

**EXERCISE:** Prove that a supercommutator of superderivations is again a superderivation.
Lie derivative

**DEFINITION:** Let $B$ be a smooth manifold, and $v \in TM$ a vector field. An endomorphism $\text{Lie}_v : \Lambda^* M \longrightarrow \Lambda^* M$, preserving the grading is called a **Lie derivative along** $v$ if it satisfies the following conditions.

1. On functions $\text{Lie}_v$ is equal to a derivative along $v$.  
2. $[\text{Lie}_v, d] = 0$.  
3. $\text{Lie}_v$ is a derivation of the de Rham algebra.

**REMARK:** The algebra $\Lambda^*(M)$ is generated by $C^\infty M = \Lambda^0(M)$ and $d(C^\infty M)$. The restriction $\text{Lie}_v |_{C^\infty M}$ is determined by the first axiom. On $d(C^\infty M)$ is also determined because $\text{Lie}_v(df) = d(\text{Lie}_v f)$. **Therefore,** $\text{Lie}_v$ is uniquely defined by these axioms.

**LEMMA:** $\{d, \{d, E\}\} = 0$ for each $E \in \text{End}(\Lambda^* M)$.

**Proof:** By the super Jacobi identity, $\{d, \{d, E\}\} = -\{d, \{d, E\}\} + \{\{d, d\}, E\}$, however, $\{d, d\} = 2d^2 = 0$. ■

**THEOREM:** (Cartan's formula) Let $i_v$ be a convolution with a vector field. Then $\{d, i_v\}$ is a Lie derivative along $v$.

**Proof:** $\{d, \{d, i_v\}\} = 0$ by the lemma above. A supercommutator of two derivations is a derivation. Finally, $\{d, i_v\}$ acts on functions as $i_v(df) = \langle v, df \rangle$. ■