Geometry of manifolds

Lecture 10: de Rham algebra

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Kähler differentials

DEFINITION: Let R be a ring over a field k, and V an R-module. A k-linear map $D: R \longrightarrow V$ is called **a derivation** if it satisfies **the Leibnitz identity** $D(ab) = aD(b) + bD(a)$. The space of derivations from R to V is denoted $\operatorname{Der}_k(R, V)$.

REMARK: $\operatorname{Der}_k(R, V)$ **is an R-module**, with a natural R-action.

DEFINITION: Let R be a ring over a field k. Define an R-module $\Omega^1_k R$ **(the module of Kähler differentials)** with the following generators and relations.

* **Generators** of $\Omega^1_k R$ are indexed by elements of R; for each $a \in R$, the corresponding generator of $\Omega^1_k R$ is denoted da.
* **Relations** in $\Omega^1_k R$ are generated by expressions $d(ab) = adb + bda$, for all $a, b \in R$, and $d\lambda = 0$ for each $\lambda \in k$.

EXERCISE: Prove that the map $d: R \longrightarrow \Omega^1_k R$ mapping a to da **is a derivation**.

Universal property of Kähler differentials

CLAIM: Let V be an R-module, and $D \in \operatorname{Der}_k(R, V)$ a derivation. Then **there exists a unique R-module homomorphism $\varphi_D: \Omega^1_k R \longrightarrow V$ mapping bda to $bD(a)$.**

REMARK: Consider a category $\mathcal{C}$ of R-modules equipped with a derivation $(V, D: R \longrightarrow V)$, and define morphisms in $\mathcal{C}$ as morphisms of R-modules which commute with the derivation map. Then **$\Omega^1 R$ is an initial object in this category.** This is called **the universal property of the module of Kähler differentials**.

CLAIM: $\operatorname{Der}_k(R, V) = \operatorname{Hom}_R(\Omega^1 R, V)$.

Proof: A composition of a derivation $R \xrightarrow{d} \Omega^1 R$ and an R-module homomorphism $\Omega^1 R \longrightarrow V$ lies in $\operatorname{Der}_k(R, V)$. On the other hand, **any derivation $\xi \in \operatorname{Der}_k(R, V)$ is obtained this way,** by the universal property. ■

COROLLARY: $\operatorname{Der}_k(R) = (\Omega^1 R)^*$, where $V^* := \operatorname{Hom}(V, R)$.

Kähler differentials over polynomials

REMARK: Unlike the derivations, **the Kähler differentials are functorial on R.**

CLAIM: Let $R \xrightarrow{\varphi} R'$ be a ring homomorphism. Consider $\Omega^1 R'$ as an R-module, using the action $r, a \longrightarrow \varphi(r)a$. Then **there exists an R-module homomorphism $\Omega^1 R \longrightarrow \Omega^1 R'$, mapping dr to $d\varphi(r)$.**

CLAIM: Let $R = k[t_1, ..., t_n]$ be a polynomial ring over a field of characteristic 0. **Then $\Omega^1_k R$ is a free R-module generated by $dt_1, dt_2, ..., dt_n$.**

Proof. Step 1: For each polynomial $P \in R$, the element dP can be expressed as a sum $\sum \frac{dP}{dt_i} dt_i$. **Therefore, each $\alpha \in \Omega^1 R$ cam be written as $\sum_i Q_i dt_i$.**

Step 2: This expression $\alpha = \sum_i Q_i dt_i$ is unique, because the pairing $\operatorname{Der}_k(R) \times \Omega^1 R \longrightarrow R$ maps $\frac{d}{dt_k} \times \sum_i Q_i dt_i$ to Q_k, hence **the coefficients $Q_k \in R$ are determined unambiguously.** ■

Cotangent bundle

DEFINITION: Let A, B be finitely generated R-modules, and $\nu: A \times B \longrightarrow R$ a bilinear pairing. Define **the annihilator of ν in B** as a submodule consisting of all elements $b \in B$ for which the homomorphism $\nu(\cdot, b): A \longrightarrow R$ vanishes.

DEFINITION: Let M be a smooth manifold, $R := C^\infty M$ the ring of smooth finctions, and $\nu: \operatorname{Der}(R) \times \Omega^1 R \longrightarrow R$ the pairing obtained from an isomorphism $\operatorname{Der}(R) = (\Omega^1 R)^*$. Consider its annihilator $K \subset \Omega^1 R$. Define **the cotangent bundle** as $\Lambda^1 M := \Omega^1 R/K$.

CLAIM: $\Lambda^1 M$ is generated as a $C^\infty M$-module by $d(C^\infty M)$.

CLAIM: In these assumptions, **$\Lambda^1 M$ is an image of the tautological map $\Omega^1 M \xrightarrow{\tau} (\Omega^1 M)^{**}$. Moreover, $\Lambda^1(M) = \operatorname{Der}(R)^*$.**

Proof. Step 1: By construction, **$\Lambda^1 M$ is a quotient of $\Omega^1 M$ by a kernel of a map $\Omega^1 M \xrightarrow{\tau} \operatorname{Der}(R)^* = (\Omega^1 M)^{**}$. Therefore, $\Lambda^1 M = \operatorname{im} \tau$.**

Step 2: Let $V \subset \operatorname{Der}(R)^*$ be a $C^\infty M$-submodule generated by the symbols $df \in \operatorname{Der}(R)^*$ which are paired with vector fields v as $\langle v, df \rangle$. **Clearly, $V = \operatorname{im} \tau$. If M admits coordinates $t_1, ..., t_n$, one has $V = \operatorname{Der}(R)^*$,** because $\operatorname{Der}(R)^*$ is freely generated by $\frac{d}{dt_i}$.

Cotangent bundle (cont.)

CLAIM: In these assumptions, $\Lambda^1 M$ **is an image of the tautological map** $\Omega^1 M \xrightarrow{\tau} (\Omega^1 M)^{**}$**. Moreover,** $\Lambda^1(M) = \operatorname{Der}(R)^*$**.**

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Step 2: If M **admits coordinates** $t_1, ..., t_n$**, one has** $V = \operatorname{Der}(R)^*$**,** because $\operatorname{Der}(R)^*$ is freely generated by $\frac{d}{dt_i}$.

Step 3: Using partition of unity, we obtain that V **is a subsheaf of** $\operatorname{Der}(R)^* = (TM)^*$**.** Indeed, suppose that for some covering $\{U_i\}$ one has $\alpha\Big|_{U_i} = \sum f_j(i) dg_j(i)$, φ_i a partition of unity subordinate to $\{U_i\}$, and ψ_i functions with support in U_i satisfying $\psi_i|_{\operatorname{sup} \varphi_i} = 1$. Then $\alpha = \sum \varphi_i f_j(i) d(\psi_i g_j(i))$.

Step 4: Locally in M**,** $V = \operatorname{Der}(R)^*$**,** because locally M admits coordinates. Therefore, the sheaves V and $\operatorname{Der}(R)^*$ coincide. ■

COROLLARY: $\Lambda^1 M$ **is a locally free sheaf of** $C^\infty M$**-modules. Moreover,** $\Lambda^1 M = TM^*$**.**

COROLLARY: $\Lambda^1 M$ **is generated as a** $C^\infty M$**-module by** $d(C^\infty M)$**.**

De Rham algebra

DEFINITION: Let M be a smooth manifold. **A bundle of differential i-forms on M** is the bundle $\Lambda^i T^*M$ of antisymmetric i-forms on TM. It is denoted $\Lambda^i M$.

REMARK: $\Lambda^0 M = C^\infty M$.

DEFINITION: Let $\alpha \in (V^*)^{\otimes i}$ and $\alpha \in (V^*)^{\otimes j}$ be polylinear forms on V. Define the **tensor multiplication** $\alpha \otimes \beta$ as

$$\alpha \otimes \beta(x_1, ..., x_{i+j}) := \alpha(x_1, ..., x_j)\beta(x_{i+1}, ..., x_{i+j}).$$

DEFINITION: Let $\bigotimes_k T^*M \stackrel{\Pi}{\longrightarrow} \Lambda^k M$ be the antisymmetrization map,

$$\Pi(\alpha)(x_1, ..., x_n) := \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}_n} (-1)^\sigma \alpha(x_{\sigma_1}, x_{\sigma_2}, ..., x_{\sigma_n}).$$

Define **the exterior multiplication** $\wedge : \Lambda^i M \times \Lambda^j M \longrightarrow \Lambda^{i+j} M$ as $\alpha \wedge \beta := \Pi(\alpha \otimes \beta)$, where $\alpha \otimes \beta$ is a section $\Lambda^i M \otimes \Lambda^j M \subset \bigotimes_{i+j} T^*M$ obtained as their tensor multiplication.

REMARK: The fiber of the bundle $\Lambda^* M$ at $x \in M$ **is identified with the Grassmann algebra $\Lambda^* T^*_x M$.** This identification is compatible with the Grassmann product.

Coordinate monomials

DEFINITION: Let $t_1, ..., t_n$ be coordinate functions on $\mathbb{R}^n$, and $\alpha \in \Lambda^* \mathbb{R}^n$ a monomial obtained as a product of several dt_i: $\alpha = dt_{i_1} \wedge dt_{i_2} \wedge ... \wedge dt_{i_k}$ $i_1 < i_2 < ... < i_k$. Then α is called **a coordinate monomial**.

CLAIM: **$\Lambda^* \mathbb{R}^n$ is a trivial bundle, and coordinate monomials are free generators of $\Lambda^* \mathbb{R}^n$.**

DEFINITION: An associative algebra $A^* = \oplus_{i \in \mathbb{Z}} A^i$ is called **a graded algebra** if for all $a \in A^i$, $b \in A^j$, the product ab lies in A^{i+j}.

EXAMPLE: De Rham algebra is a graded algebra.

De Rham differential

DEFINITION: De Rham differential $d: \Lambda^* M \longrightarrow \Lambda^{*+1} M$ is an $\mathbb{R}$-linear map satisfying the following conditions.

* For each $f \in \Lambda^0 M = C^\infty M$, $d(f) \in \Lambda^1 M$ is equal to the image of the Kähler differential $df \in \Omega^1 M$ in $\Lambda^1 M = \Omega^1 M/K$.
* **(Leibnitz rule)** $d(a \wedge b) = da \wedge b + (-1)^j a \wedge db$ for any $a \in \Lambda^i M, b \in \Lambda^j M$.
* $d^2 = 0$.

REMARK: A map on a graded algebra which satisfies the Leibnitz rule above is called **an odd derivation**.

REMARK: The following two lemmas are needed to prove uniqueness of de Rham differential.

LEMMA: Let $A = \bigoplus A^i$ be a graded algebra, $B \subset A$ a set of multiplicative generators, and $D_1, D_2: A \longrightarrow A$ two odd derivations which are equal on B. **Then $D_1 = D_2$.** ■

LEMMA: $\Lambda^* M$ is generated by $C^\infty M$ and $d(C^\infty M)$.

Proof: By definition, $\Lambda^* M$ is generated by $\Lambda^0 M = C^\infty M$ and $\Lambda^1 M$. However, $d(C^\infty M)$ generate $\Lambda^1 M$, as shown above. ■

De Rham differential: uniqueness and existence

THEOREM:
De Rham differential is uniquely determined by these axioms.

Proof: De Rham differential is an odd derivation. Its value on $C^\infty M$ is defined by the first axiom. On $d(C^\infty M)$ de Rham differential valishes, because $d^2 = 0$.
■

DEFINITION: Let $t_1, ..., t_n$ be coordinate functions on $\mathbb{R}^n$, α_i coordinate monomials, and $\alpha := \sum f_i \alpha_i$. Define $d(\alpha) := \sum_i \sum_j \frac{df_i}{dt_j} dt_j \wedge \alpha_i$.

EXERCISE:
Check that d satisfies the properties of de Rham differential.

COROLLARY: **De Rham differential exists on any smooth manifold.**

Proof: Locally, de Rham differential d exists, as follows from the construction above. Since d is unique, it is compatible with restrictions. **This means that d defines a sheaf morphism.** Restricting this sheaf morphism to global sections, we obtain de Rham differential on $\Lambda^* M$. ■

Superalgebras

DEFINITION: Let $A^* = \oplus_{i\in\mathbb{Z}} A^i$ be a graded algebra over a field. It is called **graded commutative**, or **supercommutative**, if $ab = (-1)^{ij} ba$ for all $a \in A^i, b \in A^j$.

EXAMPLE: Grassmann algebra $\Lambda^* V$ is clearly supercommutative.

DEFINITION: Let A^* be a graded commutative algebra, and $D: A^* \longrightarrow A^{*+i}$ be a map which shifts grading by i. It is called a **graded derivation** if $D(ab) = D(a)b + (-1)^{ij} aD(b)$, for each $a \in A^j$.

REMARK: If i is even, graded derivation is a usual derivation. If it is even, it an odd derivation.

DEFINITION: Let M be a smooth manifold, and $X \in TM$ a vector field. Consider an operation of **convolution with a vector field** $i_X: \Lambda^i M \longrightarrow \Lambda^{i-1} M$, mapping an i-form α to an $(i-1)$-form $v_1, ..., v_{i-1} \longrightarrow \alpha(X, v_1, ..., v_{i-1})$

EXERCISE: Prove that i_X is an odd derivation.

Supercommutator

DEFINITION: Let A^* be a graded vector space, and $E: \ A^* \longrightarrow A^{*+i}$, $F: \ A^* \longrightarrow A^{*+j}$ operators shifting the grading by i, j. Define **the supercommutator** $\{E, F\} := EF - (-1)^{ij} FE$.

DEFINITION: An endomorphism which shifts a grading by i is called **even** if i is even, and **odd** otherwise.

EXERCISE: Prove that a supercommutator satisfies **graded Jacobi identity,**

$$\{E, \{F, G\}\} = \{\{E, F\}, G\} + (-1)^{\tilde{E}\tilde{F}} \{F, \{E, G\}\}$$

where $\tilde{E}$ and $\tilde{F}$ are 0 if E, F are even, and 1 otherwise.

REMARK: There is a simple mnemonic rule which allows one to remember a superidentity, if you know the commutative analogue. Each time when in commutative case two letters A, F are exchanged, in supercommutative case one needs to multiply by $(-1)^{\tilde{E}\tilde{F}}$.

EXERCISE: **Prove that a supercommutator of superderivations is again a superderivation.**

Lie derivative

DEFINITION: Let B be a smooth manifold, and $v \in TM$ a vector field. An endomorphism $\operatorname{Lie}_v : \Lambda^* M \longrightarrow \Lambda^* M$, preserving the grading is called **a Lie derivative along v** if it satisfies the following conditions.

(1) On functions Lie_v is equal to a derivative along v. (2) $[\operatorname{Lie}_v, d] = 0$.

(3) Lie_v is a derivation of the de Rham algebra.

REMARK: The algebra $\Lambda^*(M)$ is generated by $C^\infty M = \Lambda^0(M)$ and $d(C^\infty M)$. The restriction $\operatorname{Lie}_v |_{C^\infty M}$ is determined by the first axiom. On $d(C^\infty M)$ is also determined because $\operatorname{Lie}_v(df) = d(\operatorname{Lie}_v f)$. **Therefore, Lie_v is uniquely defined by these axioms.**

LEMMA: $\{d, \{d, E\}\} = 0$ **for each** $E \in \operatorname{End}(\Lambda^* M)$**.**

Proof: By the super Jacobi identity, $\{d, \{d, E\}\} = -\{d, \{d, E\}\} + \{\{d, d, \}E\}\}$, however, $\{d, d\} = 2d^2 = 0$. ∎

THEOREM: **(Cartan's formula)** Let i_v be a convolution with a vector field. **Then $\{d, i_v\}$ is a Lie derivative along v.**

Proof: $\{d, \{d, i_v\}\} = 0$ by the lemma above. A supercommutator of two derivations is a derivation. Finally, $\{d, i_v\}$ acts on functions as $i_v(df) = \langle v, df\rangle$.
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