Geometry of manifolds

Lecture 12: Poincaré lemma

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De Rham algebra (reminder)

DEFINITION: Let M be a smooth manifold. **A bundle of differential i-forms on M** is the bundle $\Lambda^i T^*M$ of antisymmetric i-forms on TM. It is denoted $\Lambda^i M$.

REMARK: $\Lambda^0 M = C^\infty M$.

DEFINITION: Let $\alpha \in (V^*)^{\otimes i}$ and $\alpha \in (V^*)^{\otimes j}$ be polylinear forms on V. Define the **tensor multiplication** $\alpha \otimes \beta$ as

$$\alpha \otimes \beta(x_1, ..., x_{i+j}) := \alpha(x_1, ..., x_j)\beta(x_{i+1}, ..., x_{i+j}).$$

DEFINITION: Let $\bigotimes_k T^*M \xrightarrow{\Pi} \Lambda^k M$ be the antisymmetrization map,

$$\Pi(\alpha)(x_1, ..., x_n) := \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}_n} (-1)^\sigma \alpha(x_{\sigma_1}, x_{\sigma_2}, ..., x_{\sigma_n}).$$

Define **the exterior multiplication** $\wedge : \Lambda^i M \times \Lambda^j M \longrightarrow \Lambda^{i+j} M$ as $\alpha \wedge \beta := \Pi(\alpha \otimes \beta)$, where $\alpha \otimes \beta$ is a section $\Lambda^i M \otimes \Lambda^j M \subset \bigotimes_{i+j} T^*M$ obtained as their tensor multiplication.

REMARK: The fiber of the bundle $\Lambda^* M$ at $x \in M$ **is identified with the Grassmann algebra $\Lambda^* T_x^* M$.** This identification is compatible with the Grassmann product.

De Rham differential (reminder)

DEFINITION: De Rham differential $d: \Lambda^* M \longrightarrow \Lambda^{*+1} M$ is an $\mathbb{R}$-linear map satisfying the following conditions.

* For each $f \in \Lambda^0 M = C^\infty M$, $d(f) \in \Lambda^1 M$ is equal to the image of the Kähler differential $df \in \Omega^1 M$ in $\Lambda^1 M = \Omega^1 M/K$.
* **(Leibnitz rule)** $d(a \wedge b) = da \wedge b + (-1)^j a \wedge db$ for any $a \in \Lambda^i M, b \in \Lambda^j M$.
* $d^2 = 0$.

THEOREM:
De Rham differential is uniquely determined by these axioms.

REMARK: The proof of uniqueness is based on the following lemmas.

LEMMA: Let $A = \bigoplus A^i$ be a graded algebra, $B \subset A$ a set of multiplicative generators, and $D_1, D_2: A \longrightarrow A$ two odd derivations which are equal on B. **Then $D_1 = D_2$.** ■

LEMMA: $\Lambda^* M$ is generated by $C^\infty M$ and $d(C^\infty M)$.

REMARK: Let $t_1, ..., t_n$ be coordinate functions on $\mathbb{R}^n$, α_i coordinate monomials, and $\alpha := \sum f_i \alpha_i$. Define $d(\alpha) := \sum_i \sum_j \frac{df_i}{dt_j} dt_j \wedge \alpha_i$. **Then d satisfies axioms of de Rham differential.** This proves its existence.

Lie derivative (reminder)

DEFINITION: Let B be a smooth manifold, and $v \in TM$ a vector field. An endomorphism $\operatorname{Lie}_v : \Lambda^* M \longrightarrow \Lambda^* M$, preserving the grading is called **a Lie derivative along v** if it satisfies the following conditions.

(1) On functions Lie_v is equal to a derivative along v. (2) $[\operatorname{Lie}_v, d] = 0$.

(3) Lie_v is a derivation of the de Rham algebra.

REMARK: The algebra $\Lambda^*(M)$ is generated by $C^\infty M = \Lambda^0(M)$ and $d(C^\infty M)$. The restriction $\operatorname{Lie}_v |_{C^\infty M}$ is determined by the first axiom. On $d(C^\infty M)$ is also determined because $\operatorname{Lie}_v(df) = d(\operatorname{Lie}_v f)$. **Therefore, Lie_v is uniquely defined by these axioms.**

LEMMA: $\{d, \{d, E\}\} = 0$ **for each** $E \in \operatorname{End}(\Lambda^* M)$.

Proof: By the super Jacobi identity, $\{d, \{d, E\}\} = -\{d, \{d, E\}\} + \{\{d, d, \} E\}\}$, however, $\{d, d\} = 2d^2 = 0$. ■

THEOREM: (Cartan's formula) Let i_v be a convolution with a vector field. **Then $\{d, i_v\}$ is a Lie derivative along v.**

Proof: $\{d, \{d, i_v\}\} = 0$ by the lemma above. A supercommutator of two derivations is a derivation. Finally, $\{d, i_v\}$ acts on functions as $i_v(df) = \langle v, df \rangle$. ■

Pullback of a differential form (reminder)

DEFINITION: Let $M \xrightarrow{\varphi} N$ be a morphism of smooth manifolds, and $\alpha \in \Lambda^i N$ be a differential form. Consider an i-form $\varphi^*\alpha$ taking value

$$\alpha\Big|_{\varphi(m)}(D_\varphi(x_1), ... D_\varphi(x_i))$$

on $x_1, ..., x_i \in T_m M$. It is called **the pullback of α**. If $M \xrightarrow{\varphi} N$ is a closed embedding, the form $\varphi^*\alpha$ is called **the restriction** of α to $M \hookrightarrow N$.

LEMMA: (*) Let $\Psi_1, \Psi_2 : \Lambda^* N \longrightarrow \Lambda^* M$ be two maps which satisfy graded Leibnitz identity, commute with de Rham differential, and satisfy $\Psi_1|_{C^\infty M} = \Psi_2|_{C^\infty M}$. **Then $\Psi_1 = \Psi_2$.**

Proof: The algebra $\Lambda^* M$ is generated multiplicatively by $C^\infty M$ and $d(C^\infty M)$; restrictions of Ψ_i to these two spaces are equal. ■

CLAIM: Pullback commutes with the de Rham differential.

Proof: Follows from Lemma (*). ■

Flow of diffeomorphisms (reminder)

DEFINITION: Let $f: M\times [a,b]\longrightarrow M$ be a smooth map such that for all $t\in[a,b]$ the restriction $f_t := f\Big|_{M\times\{t\}}: M\longrightarrow M$ is a diffeomorphism. Then f is called **a flow of diffeomorphisms**.

CLAIM: Let V_t be a flow of diffeomorphisms, $f\in C^\infty M$, and $V_t^*(f)(x) := f(V_t(x))$. Consider the map $\frac{d}{dt}V_t|_{t=c}: C^\infty M\longrightarrow C^\infty M$, with $\frac{d}{dt}V_t|_{t=c}(f) = (V_c^{-1})^*\frac{dV_t}{dt}|_{t=c} f$. **Then $(V_c^{-1})^*\frac{d}{dt}V_t|_{t=c}$ is a derivation** (that is, a vector field).

Proof:

$$\frac{d}{dt}V_t|_{t=c}(fg) = V_c^* f\frac{d}{dt}V_t|_{t=c}(g) + V_c^* g\frac{d}{dt}V_t|_{t=c}(f).$$

■

DEFINITION: The vector field $(V_c^{-1})^*\frac{d}{dt}V_t|_{t=c}$ is called **a vector field tangent to a flow of diffeomorphisms V_t at $t=c$.**

Lie derivative and a flow of diffeomorphisms (reminder)

DEFINITION: Let v be a vector field on M, and $V: M\times[a,b]\longrightarrow M$ a flow of diffeomorphisms which satisfies $(V_c^{-1})^*\frac{d}{dt}V_t|_{t=c}=v$ for each c, and $V_0=\mathrm{Id}$. Then V_t is called **an exponent of** v.

CLAIM: Exponent of a vector field is unique; it exists when M is compact. This statement is called **"Picard-Lindelöf theorem"** or **"uniqueness and existence of solutions of ordinary differential equations"**.

PROPOSITION: Let v be a vector field, and V_t its exponent. For any $\alpha\in\Lambda^*M$, consider $V_t^*\alpha$ as a Λ^*M-valued function of t. **Then** $\mathrm{Lie}_v\alpha=\frac{d}{dt}(V_t^*\alpha)|_{t=0}$**.**

Proof: By definition, $\mathrm{Lie}_v=\frac{d}{dt}V_t$ on functions. Lie_v commutes with de Rham differential, because $\mathrm{Lie}_v=i_vd+di_v$. The map $\frac{d}{dt}V_t$ commutes with de Rham differential, because it is a derivative of a pullback. Now **Lemma (*) is applied to show that** $\mathrm{Lie}_v\alpha=\frac{d}{dt}(V_t^*\alpha)$**.** ■

Homotopy operators

DEFINITION: A complex is a sequence of vector spaces and homomorphisms ... $\xrightarrow{d} C_{i-1} \xrightarrow{d} C_i \xrightarrow{d} C_{i+1} \xrightarrow{d}$ ...
satisfying $d^2 = 0$. **Homomorphism** $(C_*, d) \longrightarrow (C'_*, d)$ of complexes is a sequence of homomorphism $C_i \longrightarrow C'_i$ commuting with the differentials.

DEFINITION: An element $c \in C_i$ is called **closed** if $c \in \ker d$ and **exact** if $c \in \operatorname{im} d$. **Cohomology** of a complex is a quotient $\frac{\ker d}{\operatorname{im} d}$.

REMARK: A homomorphism of complexes induces a natural homomorphism of cohomology groups.

DEFINITION: Let (C_*, d), (C'_*, d) be a complex. **Homotopy** is a sequence of maps $h: C_* \longrightarrow C'_{*-1}$. Two homomorphisms $f, g: (C_*, d) \longrightarrow (C'_*, d)$ are called **homotopy equivalent** if $f - g = \{h, d\}$ for some homotopy operator h.

CLAIM: Let $f, f': (C_*, d) \longrightarrow (C'_*, d)$ be homotopy equivalent maps of complexes. **Then f and f' induce the same maps on cohomology.**

Proof. Step 1: Let $g := f - f'$. It would suffice to prove that g induces 0 on cohomology.

Lie derivative and homotopy

CLAIM: Let $f, f' : (C_*, d) \longrightarrow (C'_*, d)$ be homotopy equivalent maps of complexes. **Then f and f' induce the same maps on cohomology.**

Proof. Step 1: Let $g := f - f'$. It would suffice to prove that g induces 0 on cohomology.

Step 2: Let $c \in C_i$ be a closed element. **Then** $g(c) = dh(c) + hd(c) = dh(c)$ **exact.** ■

DEFINITION: Let d be de Rham differential. A form in $\ker d$ is called **closed**, a form in $\operatorname{im} d$ is called **exact**. Since $d^2 = 0$, any exact form is closed. **The group of i-th de Rham cohomology of M**, denoted $H^i(M)$, is a quotient of a space of closed i-forms by the exact: $H^*(M) = \frac{\ker d}{\operatorname{im} d}$.

REMARK: Let v be a vector field, and $\operatorname{Lie}_v : \Lambda^* M \longrightarrow \Lambda^* M$ be the corresponding Lie derivative. Then Lie_v **commutes with the de Rham differential, and acts trivially on the de Rham cohomology.**

Proof: $\operatorname{Lie}_v = i_v d + d i_v$ maps closed forms to exact. ■

Poincaré lemma

DEFINITION: An open subset $U \subset \mathbb{R}^n$ is called **starlike** if for any $x \in U$ the interval $[0, x]$ belongs to U.

THEOREM: (Poicaré lemma) Let $U \subset \mathbb{R}^n$ be a starlike subset. **Then $H^i(U) = 0$ for $i > 0$.**

REMARK: The proof would follow if we construct a vector field $\vec{r}$ such that $\operatorname{Lie}_{\vec{r}}$ is invertible on $\Lambda^*(M)$: $\operatorname{Lie}_{\vec{r}} R = \operatorname{Id}$. Indeed, for any closed form α we would have $\alpha = \operatorname{Lie}_{\vec{r}} R\alpha = di_{\vec{r}} R\alpha + i_{\vec{r}} R d\alpha = di_{\vec{r}} R\alpha$, hence any closed form is exact.

Then Poincaré lemma is implied by the following statement.

PROPOSITION: Let $U \subset \mathbb{R}^n$ be a starlike subset, $t_1, ..., t_n$ coordinate functions, and $\vec{r} := \sum t_i \frac{d}{dt_i}$ the radial vector field. **Then $\operatorname{Lie}_{\vec{r}}$ is invertible on $\Lambda^i(U)$ for $i > 0$.**

Radial vector field on starlike sets

PROPOSITION: Let $U \subset \mathbb{R}^n$ be a starlike subset, $t_1, ..., t_n$ coordinate functions, and $\vec{r} := \sum t_i \frac{d}{dt_i}$ the radial vector field. **Then $\operatorname{Lie}_{\vec{r}}$ is invertible on $\Lambda^i(U)$ for $i > 0$.**

Proof. Step 1: Let t be the coordinate function on a real line, $f(t) \in C^\infty\mathbb{R}$ a smooth function, and $v := t\frac{d}{dt}$ a vector field. Define $R(f)(t) := \int_0^1 \frac{f(\lambda t)}{\lambda} d\lambda$. Then this integral converges whenever $f(0) = 0$, and satisfies $\operatorname{Lie}_v R(f) = f$. Indeed,

$$\int_0^1 \frac{f(\lambda t)}{\lambda} d\lambda = \int_0^t \frac{f(\lambda t)}{t\lambda} d(t\lambda) = \int_0^t \frac{f(z)}{z} d(z),$$

hence $\operatorname{Lie}_v R(f) = t\frac{f(t)}{t} = f(t)$.

Step 2: Consider a function $f \in C^\infty\mathbb{R}^n$ satisfying $f(0) = 0$, and $x = (x_1, ..., x_n) \in \mathbb{R}^n$. **Then**

$$R(f)(x) := \int_0^1 \frac{f(\lambda x)}{\lambda} d\lambda$$

converges, and satisfies $\operatorname{Lie}_{\vec{r}} R(f) = f$.

Radial vector field on starlike sets (cont.)

Step 3: Consider a differential form $\alpha \in \Lambda^i$, and let $h_\lambda x \longrightarrow \lambda x$ be the homothety with coefficient $\lambda \in [0,1]$. Define

$$R(\alpha) := \int_0^1 \lambda^{-1} h_\lambda^*(\alpha) d\lambda.$$

Since $h_\lambda^*(\alpha) = 0$ for $\lambda = 0$, this integral converges. **It remains to prove that $\operatorname{Lie}_{\vec{r}} R = \operatorname{Id}$.**

Step 4: Let α be a coordinate monomial, $\alpha = dt_{i_1} \wedge dt_{i_2} \wedge ... \wedge dt_{i_k}$. Clearly, $\operatorname{Lie}_{\vec{r}}(T^{-1}\alpha) = 0$, where $T = t_{i_1} t_{i_2} ... t_{i_k}$. **Since $h_\lambda^*(f\alpha) = h_\lambda^*(tf)T^{-1}\alpha$, we have $R(f\alpha) = R(Tf)T^{-1}\alpha$ for any function $f \in C^\infty M$.** This gives

$$\operatorname{Lie}_{\vec{r}} R(f\alpha) = \operatorname{Lie}_{\vec{r}} R(Tf)T^{-1}\alpha = TfT^{-1}\alpha = f\alpha.$$

■