Geometry of manifolds

Lecture 12: Poincaré lemma

Misha Verbitsky

Math in Moscow and HSE

May 13, 2013
De Rham algebra (reminder)

**DEFINITION:** Let $M$ be a smooth manifold. A **bundle of differential $i$-forms on $M$** is the bundle $\Lambda^i T^* M$ of antisymmetric $i$-forms on $TM$. It is denoted $\Lambda^i M$.

**REMARK:** $\Lambda^0 M = C^\infty M$.

**DEFINITION:** Let $\alpha \in (V^*)^i$ and $\alpha \in (V^*)^j$ be polylinear forms on $V$. Define the **tensor multiplication** $\alpha \otimes \beta$ as
\[
\alpha \otimes \beta(x_1, \ldots, x_{i+j}) := \alpha(x_1, \ldots, x_j) \beta(x_{i+1}, \ldots, x_{i+j}).
\]

**DEFINITION:** Let $\otimes_k T^* M \overset{\Pi}{\longrightarrow} \Lambda^k M$ be the antisymmetrization map,
\[
\Pi(\alpha)(x_1, \ldots, x_n) := \frac{1}{n!} \sum_{\sigma \in \text{Sym}_n} (-1)^\sigma \alpha(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}).
\]

Define the **exterior multiplication** $\wedge : \Lambda^i M \times \Lambda^j M \rightarrow \Lambda^{i+j} M$ as $\alpha \wedge \beta := \Pi(\alpha \otimes \beta)$, where $\alpha \otimes \beta$ is a section $\Lambda^i M \otimes \Lambda^j M \subset \otimes_{i+j} T^* M$ obtained as their tensor multiplication.

**REMARK:** The fiber of the bundle $\Lambda^* M$ at $x \in M$ is identified with the **Grassmann algebra** $\Lambda^* T^*_x M$. This identification is compatible with the Grassmann product.
De Rham differential (reminder)

**DEFINITION:** **De Rham differential** \( d : \Lambda^* M \rightarrow \Lambda^{*+1} M \) is an \( \mathbb{R} \)-linear map satisfying the following conditions.

* For each \( f \in \Lambda^0 M = C^\infty M \), \( d(f) \in \Lambda^1 M \) is equal to the image of the Kähler differential \( df \in \Omega^1 M \) in \( \Lambda^1 M = \Omega^1 M/K \).

* **(Leibnitz rule)** \( d(a \wedge b) = da \wedge b + (-1)^j a \wedge db \) for any \( a \in \Lambda^i M, b \in \Lambda^j M \).

* \( d^2 = 0 \).

**THEOREM:**
De Rham differential is uniquely determined by these axioms.

**REMARK:** The proof of uniqueness is based on the following lemmas.

**LEMMA:** Let \( A = \bigoplus A^i \) be a graded algebra, \( B \subset A \) a set of multiplicative generators, and \( D_1, D_2 : A \rightarrow A \) two odd derivations which are equal on \( B \). Then \( D_1 = D_2 \). ■

**LEMMA:** \( \Lambda^* M \) is generated by \( C^\infty M \) and \( d(C^\infty M) \).

**REMARK:** Let \( t_1, \ldots, t_n \) be coordinate functions on \( \mathbb{R}^n \), \( \alpha_i \) coordinate monomials, and \( \alpha := \sum f_i \alpha_i \). Define \( d(\alpha) := \sum_i \sum_j \frac{df_i}{dt_j} dt_j \wedge \alpha_i \). Then \( d \) satisfies axioms of de Rham differential. This proves its existence.
Lie derivative (reminder)

**DEFINITION:** Let $B$ be a smooth manifold, and $v \in TM$ a vector field. An endomorphism $\text{Lie}_v : \Lambda^* M \to \Lambda^* M$, preserving the grading is called a **Lie derivative along** $v$ if it satisfies the following conditions.

1. On functions $\text{Lie}_v$ is equal to a derivative along $v$. 
2. $[\text{Lie}_v, d] = 0$.
3. $\text{Lie}_v$ is a derivation of the de Rham algebra.

**REMARK:** The algebra $\Lambda^*(M)$ is generated by $\mathcal{C}^\infty M = \Lambda^0(M)$ and $d(\mathcal{C}^\infty M)$. The restriction $\text{Lie}_v |_{\mathcal{C}^\infty M}$ is determined by the first axiom. On $d(\mathcal{C}^\infty M)$ is also determined because $\text{Lie}_v(df) = d(\text{Lie}_v f)$. **Therefore, $\text{Lie}_v$ is uniquely defined by these axioms.**

**LEMMA:** $\{d, \{d, E\}\} = 0$ for each $E \in \text{End}(\Lambda^* M)$.

**Proof:** By the super Jacobi identity, $\{d, \{d, E\}\} = -\{d, \{d, E\}\} + \\{\{d, d\}, E\}$, however, $\{d, d\} = 2d^2 = 0$. ■

**THEOREM:** (Cartan's formula) Let $i_v$ be a convolution with a vector field. Then $\{d, i_v\}$ is a Lie derivative along $v$.

**Proof:** $\{d, \{d, i_v\}\} = 0$ by the lemma above. A supercommutator of two derivations is a derivation. Finally, $\{d, i_v\}$ acts on functions as $i_v(df) = \langle v, df \rangle$. ■
Pullback of a differential form (reminder)

**DEFINITION:** Let $M \xrightarrow{\varphi} N$ be a morphism of smooth manifolds, and $\alpha \in \Lambda^i N$ be a differential form. Consider an $i$-form $\varphi^* \alpha$ taking value

$$\alpha|_{\varphi(m)}(D\varphi(x_1),...D\varphi(x_i))$$

on $x_1,...,x_i \in T_m M$. It is called the pullback of $\alpha$. If $M \xrightarrow{\varphi} N$ is a closed embedding, the form $\varphi^* \alpha$ is called the restriction of $\alpha$ to $M \hookrightarrow N$.

**LEMMA:** (*) Let $\Psi_1, \Psi_2 : \Lambda^* N \longrightarrow \Lambda^* M$ be two maps which satisfy graded Leibnitz identity, commute with de Rham differential, and satisfy $\Psi_1|_{C^\infty M} = \Psi_2|_{C^\infty M}$. Then $\Psi_1 = \Psi_2$.

**Proof:** The algebra $\Lambda^* M$ is generated multiplicatively by $C^\infty M$ and $d(C^\infty M)$; restrictions of $\Psi_i$ to these two spaces are equal. ■

**CLAIM:** Pullback commutes with the de Rham differential.

**Proof:** Follows from Lemma (*). ■
Flow of diffeomorphisms (reminder)

**DEFINITION:** Let $f : M \times [a, b] \rightarrow M$ be a smooth map such that for all $t \in [a, b]$ the restriction $f_t := f|_{M \times \{t\}} : M \rightarrow M$ is a diffeomorphism. Then $f$ is called a **flow of diffeomorphisms**.

**CLAIM:** Let $V_t$ be a flow of diffeomorphisms, $f \in C^\infty M$, and $V_t^*(f)(x) := f(V_t(x))$. Consider the map $\frac{d}{dt}V_t|_{t=c} : C^\infty M \rightarrow C^\infty M$, with $\frac{d}{dt}V_t|_{t=c}(f) = (V_c^{-1})^*\frac{dV_t}{dt}|_{t=c}f$. Then $(V_c^{-1})^*\frac{d}{dt}V_t|_{t=c}$ is a **derivation** (that is, a vector field).

**Proof:**

$$\frac{d}{dt}V_t|_{t=c}(fg) = V_c^*f\frac{d}{dt}V_t|_{t=c}(g) + V_c^*g\frac{d}{dt}V_t|_{t=c}(f).$$

**DEFINITION:** The vector field $(V_c^{-1})^*\frac{d}{dt}V_t|_{t=c}$ is called a **vector field tangent to a flow of diffeomorphisms** $V_t$ at $t = c$. 
**Lie derivative and a flow of diffeomorphisms (reminder)**

**DEFINITION:** Let $v$ be a vector field on $M$, and $V : M \times [a, b] \rightarrow M$ a flow of diffeomorphisms which satisfies $(V^{-1}_c)^* \frac{d}{dt} V_t|_{t=c} = v$ for each $c$, and $V_0 = \text{Id}$. Then $V_t$ is called an exponent of $v$.

**CLAIM:** Exponent of a vector field is unique; it exists when $M$ is compact. This statement is called “Picard-Lindelöf theorem” or “uniqueness and existence of solutions of ordinary differential equations”.

**PROPOSITION:** Let $v$ be a vector field, and $V_t$ its exponent. For any $\alpha \in \Lambda^*M$, consider $V_t^*\alpha$ as a $\Lambda^*M$-valued function of $t$. Then $\text{Lie}_v \alpha = \frac{d}{dt}(V_t^*\alpha)|_{t=0}$.

**Proof:** By definition, $\text{Lie}_v = \frac{d}{dt}V_t$ on functions. $\text{Lie}_v$ commutes with de Rham differential, because $\text{Lie}_v = i_v d + d i_v$. The map $\frac{d}{dt}V_t$ commutes with de Rham differential, because it is a derivative of a pullback. Now Lemma (*) is applied to show that $\text{Lie}_v \alpha = \frac{d}{dt}(V_t^*\alpha)$. ■
Homotopy operators

**DEFINITION:** A complex is a sequence of vector spaces and homomorphisms ... \(d \rightarrow C_{i-1} \rightarrow C_i \rightarrow C_{i+1} \rightarrow \ldots\) satisfying \(d^2 = 0\). Homomorphism \((C_*, d) \rightarrow (C_*', d)\) of complexes is a sequence of homomorphism \(C_i \rightarrow C_i'\) commuting with the differentials.

**DEFINITION:** An element \(c \in C_i\) is called closed if \(c \in \ker d\) and exact if \(c \in \im d\). Cohomology of a complex is a quotient \(\ker d / \im d\).

**REMARK:** A homomorphism of complexes induces a natural homomorphism of cohomology groups.

**DEFINITION:** Let \((C_*, d), (C_*', d)\) be a complex. Homotopy is a sequence of maps \(h : C_* \rightarrow C'_{*-1}\). Two homomorphisms \(f, g : (C_*, d) \rightarrow (C_*', d)\) are called homotopy equivalent if \(f - g = \{h, d\}\) for some homotopy operator \(h\).

**CLAIM:** Let \(f, f' : (C_*, d) \rightarrow (C_*', d)\) be homotopy equivalent maps of complexes. Then \(f\) and \(f'\) induce the same maps on cohomology.

**Proof. Step 1:** Let \(g := f - f'\). It would suffice to prove that \(g\) induces 0 on cohomology.
Lie derivative and homotopy

CLAIM: Let \( f, f' : (C_*, d) \to (C'_*, d) \) be homotopy equivalent maps of complexes. Then \( f \) and \( f' \) induce the same maps on cohomology.

Proof. Step 1: Let \( g := f - f' \). It would suffice to prove that \( g \) induces 0 on cohomology.

Step 2: Let \( c \in C_i \) be a closed element. Then \( g(c) = dh(c) + hd(c) = dh(c) \) exact. □

DEFINITION: Let \( d \) be de Rham differential. A form in \( \ker d \) is called closed, a form in \( \im d \) is called exact. Since \( d^2 = 0 \), any exact form is closed. The group of \( i \)-th de Rham cohomology of \( M \), denoted \( H^i(M) \), is a quotient of a space of closed \( i \)-forms by the exact: \( H^*(M) = \frac{\ker d}{\im d} \).

REMARK: Let \( v \) be a vector field, and \( \text{Lie}_v : \Lambda^* M \to \Lambda^* M \) be the corresponding Lie derivative. Then \( \text{Lie}_v \) commutes with the de Rham differential, and acts trivially on the de Rham cohomology.

Proof: \( \text{Lie}_v = i_v d + d i_v \) maps closed forms to exact. □
Poincaré lemma

**DEFINITION:** An open subset $U \subset \mathbb{R}^n$ is called *starlike* if for any $x \in U$ the interval $[0,x]$ belongs to $U$.

**THEOREM:** *(Poincaré lemma)* Let $U \subset \mathbb{R}^n$ be a starlike subset. Then $H^i(U) = 0$ for $i > 0$.

**REMARK:** The proof would follow if we construct a vector field $\vec{r}$ such that $\text{Lie}_\vec{r}$ is invertible on $\Lambda^*(M)$: $\text{Lie}_\vec{r} R = \text{Id}$. Indeed, for any closed form $\alpha$ we would have $\alpha = \text{Lie}_\vec{r} R \alpha = d i_\vec{r} R \alpha + i_\vec{r} R d \alpha = d i_\vec{r} R \alpha$, hence any closed form is exact.

Then Poincaré lemma is implied by the following statement.

**PROPOSITION:** Let $U \subset \mathbb{R}^n$ be a starlike subset, $t_1, ..., t_n$ coordinate functions, and $\vec{r} := \sum t_i \frac{d}{dt_i}$ the radial vector field. Then $\text{Lie}_\vec{r}$ is invertible on $\Lambda^i(U)$ for $i > 0$. 
Radial vector field on starlike sets

**PROPOSITION:** Let $U \subset \mathbb{R}^n$ be a starlike subset, $t_1, \ldots, t_n$ coordinate functions, and $\vec{r} := \sum t_i \frac{d}{dt_i}$ the radial vector field. Then $\text{Lie}_\vec{r}$ is invertible on $\Lambda^i(U)$ for $i > 0$.

**Proof. Step 1:** Let $t$ be the coordinate function on a real line, $f(t) \in C^\infty \mathbb{R}$ a smooth function, and $v := t \frac{d}{dt}$ a vector field. Define $R(f)(t) := \int_0^1 \frac{f(\lambda t)}{\lambda} d\lambda$. Then this integral converges whenever $f(0) = 0$, and satisfies $\text{Lie}_v R(f) = f$. Indeed,
\[
\int_0^1 \frac{f(\lambda t)}{\lambda} d\lambda = \int_0^t \frac{f(\lambda t)}{t\lambda} d(t\lambda) = \int_0^t \frac{f(z)}{z} d(z),
\]
hence $\text{Lie}_v R(f) = t \frac{f(t)}{t} = f(t)$.

**Step 2:** Consider a function $f \in C^\infty \mathbb{R}^n$ satisfying $f(0) = 0$, and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Then
\[
R(f)(x) := \int_0^1 \frac{f(\lambda x)}{\lambda} d\lambda
\]
converges, and satisfies $\text{Lie}_\vec{r} R(f) = f$. 


Radial vector field on starlike sets (cont.)

Step 3: Consider a differential form $\alpha \in \Lambda^i$, and let $h_\lambda x \mapsto \lambda x$ be the homothety with coefficient $\lambda \in [0, 1]$. Define

$$R(\alpha) := \int_0^1 \lambda^{-1} h_\lambda^*(\alpha) d\lambda.$$ 

Since $h_\lambda^*(\alpha) = 0$ for $\lambda = 0$, this integral converges. It remains to prove that $\text{Lie}_\vec{r} R = \text{Id}$.

Step 4: Let $\alpha$ be a coordinate monomial, $\alpha = dt_{i_1} \wedge dt_{i_2} \wedge ... \wedge dt_{i_k}$. Clearly, $\text{Lie}_\vec{r} (T^{-1} \alpha) = 0$, where $T = t_{i_1} t_{i_2} ... t_{i_k}$. Since $h_\lambda^*(f\alpha) = h_\lambda^*(tf) T^{-1} \alpha$, we have $R(f\alpha) = R(Tf) T^{-1} \alpha$ for any function $f \in C^\infty M$. This gives

$$\text{Lie}_\vec{r} R(f\alpha) = \text{Lie}_\vec{r} R(Tf) T^{-1} \alpha = Tf T^{-1} \alpha = f\alpha.$$