Geometry of manifolds

lecture 2: manifolds and ringed spaces

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Sheaves and exact sequences

DEFINITION: An open cover of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$.

REMARK: A presheaf of functions is a collection of subrings of functions on open subsets, compatible with restrictions. **A sheaf of fuctions is a presheaf allowing "gluing"** a function on a bigger open set if its restrictions to smaller open sets are compatible.

DEFINITION: A sequence $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow ...$ of homomorphisms of abelian groups or vector spaces is called **exact** if the image of each map is the kernel of the next one.

CLAIM: A presheaf $\mathcal{F}$ is a sheaf if and only if for every cover $\{U_i\}$ of an open subset $U \subset M$, **the sequence of restriction maps**

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$$

is exact, with $\eta \in \mathcal{F}(U_i)$ mapped to

$$\eta\Big|_{U_i \cap U_j} - \eta\Big|_{U_j \cap U_i} \in \mathcal{F}(U_i \cap U_j) \oplus \mathcal{F}(U_j \cap U_i) \subset \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$$

Ringed spaces

A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of functions. A **morphism** $(M, \mathcal{F}) \xrightarrow{\Psi} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \xrightarrow{\Psi} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $\psi^* f := f \circ \Psi$ belongs to the ring $\mathcal{F}\Big(\Psi^{-1}(U)\Big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

DEFINITION: Let $(M, \mathcal{F})$ be a topological manifold equipped with a sheaf of functions. It is said to be a **smooth manifold of class** C^∞ or C^i if every point in $(M, \mathcal{F})$ has an open neighborhood isomorphic to the ringed space $(\mathbb{B}^n, \mathcal{F}')$, where $\mathbb{B}^n \subset \mathbb{R}^n$ is an open ball and $\mathcal{F}'$ is a ring of functions on an open ball $\mathbb{B}^n$ of this class.

DEFINITION: **Diffeomorphism** of smooth manifolds is a homeomorphism φ which induces an isomorphims of ringed spaces, that is, φ and φ^{-1} map (locally defined) smooth functions to smooth functions.

Assume from now on that all manifolds are Hausdorff and of class C^∞.

Charts and coordinates

DEFINITION: Coordinate system on a manifold M is an open subset $V \subset M$ equipped with an isomorphism of ringed spaces $(V, C^\infty V) \cong (\mathbb{B}^n, C^\infty \mathbb{B}^n)$ per definition of a manifold.

DEFINITION: A chart on a smooth manifold $(M, C^\infty M)$ is an open subset $U \subset M$ together with an embedding $\psi : U \longrightarrow \mathbb{R}^n$ given by smooth functions $\varphi_1, ..., \varphi_n \in C^\infty M$ inducing a diffeomorphism on any open subset $V \subset U$ equipped with a coordinate system $(V, C^\infty V) \cong (\mathbb{B}^n, C^\infty \mathbb{B}^n)$.

DEFINITION: Transition map between two charts $\psi_1 : U_1 \longrightarrow \mathbb{R}^n$ and $\psi_2 : U_2 \longrightarrow \mathbb{R}^n$ is a map $\Psi_{ij} : \psi_1(U_1 \cap U_2) \longrightarrow \psi_2(U_1 \cap U_2)$ defined as $\Psi_{ij}(x) = \psi_2(\psi_1^{-1}(x))$.

CLAIM: Transition maps are smooth.

Proof: In local coordinates all functions $\varphi_1, ..., \varphi_n$ used in the definition of the transition map are smooth. ■

Atlases on manifolds

DEFINITION: An atlas on a smooth manifold is a collection of charts $\{U_i, \psi_i : U_i \longrightarrow \mathbb{R}^n\}$ satisfying $\bigcup U_i = M$ together with their transition maps.

REMARK: In such a situation, the **charts U_i are usually identified with their images $\psi(U_i) \subset \mathbb{R}^n$.**

REMARK: The sheaf $C^\infty M$ can be reconstructed from an atlas as follows: a function f on $U \subset M$ is smooth if and only if its restrictions to $U \cap U_i$ are smooth on all charts.

Embedded submanifolds

DEFINITION: A **closed embedding** $\varphi : N \hookrightarrow M$ of topological spaces is an injective map from N to a closed subset $\varphi(N)$ inducing a homeomorphism of N and $\varphi(N)$.

DEFINITION: $M \subset \mathbb{R}^n$ is called **a submanifold** of dimension m if for every point $x \in N$, there is a neighborhood $U \subset \mathbb{R}^n$ diffeomorphic to an open ball, such that this diffeomorphism maps $U \cap N$ onto a linear subspace of dimension m.

DEFINITION: Let $M \subset \mathbb{R}^n$ be a submanifold. **A sheaf of smooth functions** $C^\infty M$ is defined as the sheaf of all functions on $U \subset M$ which can be locally extended to smooth functions on an open subset of $\mathbb{R}^n$ containing U.

Embedded submanifolds are smooth

EXAMPLE: Let $M = \mathbb{R}^k \subset \mathbb{R}^n$ be a linear space, and $\pi : \mathbb{R}^n \longrightarrow M$ be a linear projection. **A function f on $U \subset M$ is smooth if and only if $\pi^* f$ is smooth** (by definition, $\pi^* f(z) = f(\pi(z))$). Therefore, **any smooth function on U can be extended to $\pi^{-1}(U) \subset \mathbb{R}^n$.** We obtain that $(M, C^\infty M) \cong (\mathbb{R}^k, C^\infty \mathbb{R}^k)$.

CLAIM: Let $M \subset \mathbb{R}^n$ be a submanifold, and $C^\infty M$ the sheaf defined above. **Then $(M, C^\infty M)$ is a smooth manifold.**

Proof: Locally around each point of M, the pair $(\mathbb{R}^n, M)$ is diffeomorphic to $(\mathbb{R}^n, \mathbb{R}^k)$. Then the previous example can be applied. ∎

THEOREM: **Any manifold can be embedded to $\mathbb{R}^n$.**

Its proof takes some work and will be done in the next lecture.

Locally finite covers

DEFINITION: An open cover $\{U_\alpha\}$ of a topological space M is called **locally finite** if every point in M possesses a neighborhood that intersects only a finite number of U_α.

CLAIM: Let $\{U_\alpha\}$ be a locally finite atlas on a manifold M. **Then there exists a refinement $\{V_\beta\}$ of $\{U_\alpha\}$ such that a closure of each V_β is compact in M.**

Proof: Let $\{U_\alpha\}$ be a locally finite atlas on M, and $U_\alpha \xrightarrow{\varphi_\alpha} \mathbb{R}^n$ homeomorphisms. Consider a cover $\{V_i\}$ of $\mathbb{R}^n$ given by open balls of radius 2 centered in integer points, and let $\{W_\beta\}$ be a cover of M obtained as union of $\varphi_\alpha^{-1}(V_i)$. **Then $\{W_\beta\}$ is locally finite.** ■

DEFINITION: Let $U \subset V$ be two open subsets of M such that the closure of U is contained in V. **In this case we write $U \Subset V$.**

Locally finite covers and their subcovers

EXERCISE: Let K_1, K_2 be non-intersecting compact subsets of a Hausdorff topological space M. **Show that there exist a pair of open subsets $U_1 \supset K_1$, $U_2 \supset K_2$ satisfying $U_1 \cap U_2 = \emptyset$.**

CLAIM: Let $U \subset M$ be an open subset with compact closure, and $V \supset M \backslash U$ another open subset. **Then there exists $U' \subset U$ such that the closure of U' is contained in U, and $V \cup U' = M$.**

Proof. Step 1: The complement $M \backslash U$ does not intersect $M \backslash V$, these sets are closed, and $M \backslash V$ is compact. Replacing M by the closure $\overline{U}$, we may consider that that $M \backslash U$ is also compact.

Step 2: Take open subsets $A, B \subset M$ separating $M \backslash U$ and $M \backslash V$. Then the closure $\overline{B}$ does not intersect $M \backslash U$, because B does not intersect an open neighbourhood of $M \backslash U$. Therefore, $U' := B$ lies in U with its closure.

Step 3: By construction, $\overline{B}$ contains $M \backslash V$, hence $U' \cup V = M$. ∎

Locally finite covers and their subcovers

THEOREM: Let $\{U_\alpha\}$ be a countable locally finite cover of a Hausdorff topological space, such that a closure of each U_α is compact. **Then there exists another cover $\{V_\alpha\}$ indexed by the same set, such that $V_\alpha \Subset U_\alpha$.**

Proof. Step 1: Let $U_1, U_2, ...$ be all elements of the cover. Suppose that $V_1, ..., V_{n-1}$ is already found. To take an induction step it remains to find $V_n \Subset U_n$

Step 2: Replacing U_i by V_i and renumbering, we may assume that $n = 1$. **Then the statement of Theorem follows from the previous exercise applied to $V = \bigcup_{i=2}^\infty U_i$ and $U = U_1$.** ■

Construction of a partition of unity

REMARK: If all U_α are diffeomorphic to $\mathbb{R}^n$, all V_α can be chosen diffeomorphic to an open ball. Indeed, any compact set is contained in an open ball.

COROLLARY: Let M be a manifold admitting a locally finite countable cover $\{U_\alpha\}$, with $\varphi_\alpha : U_\alpha \longrightarrow \mathbb{R}^n$ diffeomorphisms. **Then there exists another atlas $\{U_\alpha, \varphi'_\alpha : U_\alpha \longrightarrow \mathbb{R}^n\}$, such that $\varphi'_\alpha(\mathbb{B})$ is also a cover of M, and $\mathbb{B} \subset \mathbb{R}^n$ a unit ball.** ■

EXERCISE: Find a smooth function $\nu : \mathbb{R}^n \longrightarrow [0,1]$ which vanishes outside of $\mathbb{B} \subset \mathbb{R}^n$ and is positive on $\mathbb{B}$.

REMARK: In assumptions of Corollary, let $\nu_\alpha(z) := \nu(\varphi'_\alpha)$, and $\mu_i := \frac{\nu_i}{\sum_\alpha \nu_\alpha}$. Then $\mu_\alpha : M \longrightarrow [0,1]$ are smooth functions with support in U_α satisfying $\sum_\alpha \mu_\alpha = 1$. Such a set of functions is called **a partition of unity**.

Partition of unity: a formal definition

DEFINITION: Let M be a smooth manifold and let $\{U_\alpha\}$ a locally finite cover of M. A **partition of unity** subordinate to the cover $\{U_\alpha\}$ is a family of smooth functions $f_i : M \to [0,1]$ with compact support indexed by the same indices as the U_i's and satisfying the following conditions.
(a) Every function f_i vanishes outside U_i
(b) $\sum_i f_i = 1$

The argument of previous page proves the following theorem.

THEOREM: Let $\{U_\alpha\}$ be a countable, locally finite cover of a manifold M, with all U_α diffeomorphic to $\mathbb{R}^n$. **Then there exists a partition of unity subordinate to $\{U_\alpha\}$.**

Whitney's theorem for compact manifolds

THEOREM: Let M be a compact smooth manifold. **Then M admits a closed smooth embedding to $\mathbb{R}^N$.**

Proof. Step 1: Choose a finite atlas $\{V_i, \varphi_i : V_i \longrightarrow \mathbb{R}^n, i = 1, 2, ..., m\}$, and subordinate partution of unity $\mu_i : M \longrightarrow [0,1]$. Let $\alpha : [0,1] \longrightarrow [0,1]$ be a smooth, monotonous function mapping 0 to 0 and $[1/2m, 1]$ to 1, and $\nu_i := \alpha(\mu_i)$.

Step 2: Denote by W_i the set of interior points of $\overline{W}_i := \{z \mid \nu_i(z) = 1\}$. Since $\sum_{i=1}^m \mu_i = 1$, the set $\{W_i\}$ is a cover of M.

Step 3: For each i, the map $\Phi_i(z) := \left(\nu_i\varphi_i(z), \sqrt{1 - |\varphi_i(z)|^2\nu_i(z)^2}\right)$ is injective on W_i and maps M to a sphere $S^n \subset \mathbb{R}^{n+1}$.

Step 4: The product map

$$\Psi := \prod_{i=1}^m : \Phi_i : M \longrightarrow \underbrace{S^n \times S^n \times ... \times S^n}_{m \text{ times}}$$

is an injective, continuous map from a compact, hence it is a homeomorphism to its image **(prove it)**.

Whitney's theorem for compact manifolds (cont.)

Step 5: Any smooth function on $\overline{W}_i := \{z \mid \nu_i(z) = 1\}$ can be obtained as a restriction of a smooth function on $\Phi_i(\overline{W}_i) \subset S^n$, hence **this map induces an isomorphism of the ring of smooth functions on $\overline{W}_i$ and its image.**

Step 6: Let W_i be the set of interior points of $\overline{W}_i$. We have constructed a cover $\{W_i\}$ of M such that on each W_i the map Ψ induced an isomorphism of ringed spaces. ■