# Geometry of manifolds

## lecture 2: manifolds and ringed spaces

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## Sheaves and exact sequences

**DEFINITION: An open cover** of a topological space $X$ is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$.

**REMARK: A presheaf of functions** is a collection of subrings of functions on open subsets, compatible with restrictions. **A sheaf of fuctions is a presheaf allowing "gluing"** a function on a bigger open set if its restrictions to smaller open sets are compatible.

**DEFINITION:** A sequence $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow ...$ of homomorphisms of abelian groups or vector spaces is called **exact** if the image of each map is the kernel of the next one.

**CLAIM:** A presheaf $\mathcal{F}$ is a sheaf if and only if for every cover $\{U_i\}$ of an open subset $U \subset M$, **the sequence of restriction maps**

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$$

**is exact,** with $\eta \in \mathcal{F}(U_i)$ mapped to

$$\eta\Big|_{U_i \cap U_j} - \eta\Big|_{U_j \cap U_i} \in \mathcal{F}(U_i \cap U_j) \oplus \mathcal{F}(U_j \cap U_i) \subset \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$$

# Ringed spaces

A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of functions. A **morphism** $(M, \mathcal{F}) \xrightarrow{\Psi} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \xrightarrow{\Psi} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $\psi^* f := f \circ \Psi$ belongs to the ring $\mathcal{F}\big(\Psi^{-1}(U)\big)$. An **isomorphism** of ringed spaces is a homeomorphism $\Psi$ such that $\Psi$ and $\Psi^{-1}$ are morphisms of ringed spaces.

**DEFINITION:** Let $(M, \mathcal{F})$ be a topological manifold equipped with a sheaf of functions. It is said to be a **smooth manifold of class** $C^\infty$ or $C^i$ if every point in $(M, \mathcal{F})$ has an open neighborhood isomorphic to the ringed space $(\mathbb{B}^n, \mathcal{F}')$, where $\mathbb{B}^n \subset \mathbb{R}^n$ is an open ball and $\mathcal{F}'$ is a ring of functions on an open ball $\mathbb{B}^n$ of this class.

**DEFINITION: Diffeomorphism** of smooth manifolds is a homeomorphism $\varphi$ which induces an isomorphims of ringed spaces, that is, $\varphi$ and $\varphi^{-1}$ map (locally defined) smooth functions to smooth functions.

**Assume from now on that all manifolds are Hausdorff and of class $C^\infty$.**

## Charts and coordinates

**DEFINITION: Coordinate system** on a manifold $M$ is an open subset $V \subset M$ equipped with an isomorphism of ringed spaces $(V, C^\infty V) \cong (\mathbb{B}^n, C^\infty \mathbb{B}^n)$ per definition of a manifold.

**DEFINITION: A chart** on a smooth manifold $(M, C^\infty M)$ is an open subset $U \subset M$ together with an embedding $\psi: U \longrightarrow \mathbb{R}^n$ given by smooth functions $\varphi_1, ..., \varphi_n \in C^\infty M$ inducing a diffeomorphism on any open subset $V \subset U$ equipped with a coordinate system $(V, C^\infty V) \cong (\mathbb{B}^n, C^\infty \mathbb{B}^n)$.

**DEFINITION: Transition map** between two charts $\psi_1: U_1 \longrightarrow \mathbb{R}^n$ and $\psi_2: U_2 \longrightarrow \mathbb{R}^n$ is a map $\Psi_{ij}: \psi_1(U_1 \cap U_2) \longrightarrow \psi_2(U_1 \cap U_2)$ defined as $\Psi_{ij}(x) = \psi_2(\psi_1^{-1}(x))$.

**CLAIM: Transition maps are smooth.**

**Proof:** In local coordinates all functions $\varphi_1, ..., \varphi_n$ used in the definition of the transition map are smooth. ■

## Atlases on manifolds

**DEFINITION: An atlas** on a smooth manifold is a collection of charts $\{U_i, \psi_i : \ U_i \longrightarrow \mathbb{R}^n\}$ satisfying $\bigcup U_i = M$ together with their transition maps.

**REMARK:** In such a situation, the **charts $U_i$ are usually identified with their images $\psi(U_i) \subset \mathbb{R}^n$.**

**REMARK: The sheaf $C^\infty M$ can be reconstructed from an atlas** as follows: a function $f$ on $U \subset M$ is smooth if and only if its restrictions to $U \cap U_i$ are smooth on all charts.

## Embedded submanifolds

**DEFINITION:** A **closed embedding** $\varphi: N \hookrightarrow M$ of topological spaces is an injective map from $N$ to a closed subset $\varphi(N)$ inducing a homeomorphism of $N$ and $\varphi(N)$.

**DEFINITION:** $M \subset \mathbb{R}^n$ is called **a submanifold** of dimension $m$ if for every point $x \in N$, there is a neighborhood $U \subset \mathbb{R}^n$ diffeomorphic to an open ball, such that this diffeomorphism maps $U \cap N$ onto a linear subspace of dimension $m$.

**DEFINITION:** Let $M \subset \mathbb{R}^n$ be a submanifold. **A sheaf of smooth functions** $C^\infty M$ is defined as the sheaf of all functions on $U \subset M$ which can be locally extended to smooth functions on an open subset of $\mathbb{R}^n$ containing $U$.

## Embedded submanifolds are smooth

**EXAMPLE:** Let $M = \mathbb{R}^k \subset \mathbb{R}^n$ be a linear space, and $\pi : \mathbb{R}^n \longrightarrow M$ be a linear projection. **A function $f$ on $U \subset M$ is smooth if and only if $\pi^* f$ is smooth** (by definition, $\pi^* f(z) = f(\pi(z))$). Therefore, **any smooth function on $U$ can be extended to $\pi^{-1}(U) \subset \mathbb{R}^n$.** We obtain that $(M, C^\infty M) \cong (\mathbb{R}^k, C^\infty \mathbb{R}^k)$.

**CLAIM:** Let $M \subset \mathbb{R}^n$ be a submanifold, and $C^\infty M$ the sheaf defined above. **Then $(M, C^\infty M)$ is a smooth manifold.**

**Proof:** Locally around each point of $M$, the pair $(\mathbb{R}^n, M)$ is diffeomorphic to $(\mathbb{R}^n, \mathbb{R}^k)$. Then the previous example can be applied. ■

**THEOREM: Any manifold can be embedded to $\mathbb{R}^n$.**

Its proof takes some work and will be done in the next lecture.

## Locally finite covers

**DEFINITION:** An open cover $\{U_\alpha\}$ of a topological space $M$ is called **locally finite** if every point in $M$ possesses a neighborhood that intersects only a finite number of $U_\alpha$.

**CLAIM:** Let $\{U_\alpha\}$ be a locally finite atlas on a manifold $M$. **Then there exists a refinement $\{V_\beta\}$ of $\{U_\alpha\}$ such that a closure of each $V_\beta$ is compact in $M$.**

**Proof:** Let $\{U_\alpha\}$ be a locally finite atlas on $M$, and $U_\alpha \xrightarrow{\varphi_\alpha} \mathbb{R}^n$ homeomorphisms. Consider a cover $\{V_i\}$ of $\mathbb{R}^n$ given by open balls of radius 2 centered in integer points, and let $\{W_\beta\}$ be a cover of $M$ obtained as union of $\varphi_\alpha^{-1}(V_i)$. **Then $\{W_\beta\}$ is locally finite.** ∎

**DEFINITION:** Let $U \subset V$ be two open subsets of $M$ such that the closure of $U$ is contained in $V$. **In this case we write $U \Subset V$.**

## Locally finite covers and their subcovers

**EXERCISE:** Let $K_1, K_2$ be non-intersecting compact subsets of a Hausdorff topological space $M$. **Show that there exist a pair of open subsets $U_1 \supset K_1$, $U_2 \supset K_2$ satisfying $U_1 \cap U_2 = \emptyset$.**

**CLAIM:** Let $U \subset M$ be an open subset with compact closure, and $V \supset M \backslash U$ another open subset. **Then there exists $U' \subset U$ such that the closure of $U'$ is contained in $U$, and $V \cup U' = M$.**

**Proof. Step 1:** The complement $M \backslash U$ does not intersect $M \backslash V$, these sets are closed, and $M \backslash V$ is compact. Replacing $M$ by the closure $\overline{U}$, we may consider that that $M \backslash U$ is also compact.

**Step 2:** Take open subsets $A, B \subset M$ separating $M \backslash U$ and $M \backslash V$. Then the closure $\overline{B}$ does not intersect $M \backslash U$, because $B$ does not intersect an open neighbourhood of $M \backslash U$. Therefore, $U' := B$ lies in $U$ with its closure.

**Step 3:** By construction, $\overline{B}$ contains $M \backslash V$, hence $U' \cup V = M$. ■

## Locally finite covers and their subcovers

**THEOREM:** Let $\{U_\alpha\}$ be a countable locally finite cover of a Hausdorff topological space, such that a closure of each $U_\alpha$ is compact. **Then there exists another cover $\{V_\alpha\}$ indexed by the same set, such that $V_\alpha \Subset U_\alpha$.**

**Proof. Step 1:** Let $U_1, U_2, ...$ be all elements of the cover. Suppose that $V_1, ..., V_{n-1}$ is already found. To take an induction step it remains to find $V_n \Subset U_n$

**Step 2:** Replacing $U_i$ by $V_i$ and renumbering, we may assume that $n = 1$. **Then the statement of Theorem follows from the previous exercise applied to $V = \bigcup_{i=2}^\infty U_i$ and $U = U_1$.** ■

## Construction of a partition of unity

**REMARK:** If all $U_\alpha$ are diffeomorphic to $\mathbb{R}^n$, all $V_\alpha$ can be chosen diffeomorphic to an open ball. Indeed, any compact set is contained in an open ball.

**COROLLARY:** Let $M$ be a manifold admitting a locally finite countable cover $\{U_\alpha\}$, with $\varphi_\alpha: U_\alpha \longrightarrow \mathbb{R}^n$ diffeomorphisms. **Then there exists another atlas $\{U_\alpha, \varphi'_\alpha: U_\alpha \longrightarrow \mathbb{R}^n\}$, such that $\varphi'_\alpha(\mathbb{B})$ is also a cover of $M$, and $\mathbb{B} \subset \mathbb{R}^n$ a unit ball.** ■

**EXERCISE: Find a smooth function $\nu: \mathbb{R}^n \longrightarrow [0,1]$ which vanishes outside of $\mathbb{B} \subset \mathbb{R}^n$ and is positive on $\mathbb{B}$.**

**REMARK:** In assumptions of Corollary, let $\nu_\alpha(z) := \nu(\varphi'_\alpha)$, and $\mu_i := \frac{\nu_i}{\sum_\alpha \nu_\alpha}$. Then $\mu_\alpha: M \longrightarrow [0,1]$ are smooth functions with support in $U_\alpha$ satisfying $\sum_\alpha \mu_\alpha = 1$. Such a set of functions is called **a partition of unity**.

## Partition of unity: a formal definition

**DEFINITION:** Let $M$ be a smooth manifold and let $\{U_\alpha\}$ a locally finite cover of $M$. A **partition of unity** subordinate to the cover $\{U_\alpha\}$ is a family of smooth functions $f_i : M \to [0,1]$ with compact support indexed by the same indices as the $U_i$'s and satisfying the following conditions.
(a) Every function $f_i$ vanishes outside $U_i$
(b) $\sum_i f_i = 1$

The argument of previous page proves the following theorem.

**THEOREM:** Let $\{U_\alpha\}$ be a countable, locally finite cover of a manifold $M$, with all $U_\alpha$ diffeomorphic to $\mathbb{R}^n$. **Then there exists a partition of unity subordinate to $\{U_\alpha\}$.**

## Whitney's theorem for compact manifolds

**THEOREM:** Let $M$ be a compact smooth manifold. **Then $M$ admits a closed smooth embedding to $\mathbb{R}^N$.**

**Proof. Step 1:** Choose a finite atlas $\{V_i, \varphi_i : V_i \longrightarrow \mathbb{R}^n, i = 1, 2, ..., m\}$, and subordinate partution of unity $\mu_i : M \longrightarrow [0,1]$. Let $\alpha : [0,1] \longrightarrow [0,1]$ be a smooth, monotonous function mapping 0 to 0 and $[1/2m, 1]$ to 1, and $\nu_i := \alpha(\mu_i)$.

**Step 2:** Denote by $W_i$ the set of interior points of $\overline{W}_i := \{z \mid \nu_i(z) = 1\}$. Since $\sum_{i=1}^m \mu_i = 1$, the set $\{W_i\}$ is a cover of $M$.

**Step 3:** For each $i$, the map $\Phi_i(z) := \left(\nu_i\varphi_i(z), \sqrt{1 - |\varphi_i(z)|^2\nu_i(z)^2}\right)$ is injective on $W_i$ and maps $M$ to a sphere $S^n \subset \mathbb{R}^{n+1}$.

**Step 4:** The product map

$$\Psi := \prod_{i=1}^m : \Phi_i : M \longrightarrow \underbrace{S^n \times S^n \times ... \times S^n}_{m \text{ times}}$$

is an injective, continuous map from a compact, hence it is a homeomorphism to its image **(prove it)**.

## Whitney's theorem for compact manifolds (cont.)

**Step 5:** Any smooth function on $\overline{W}_i := \{z \mid \nu_i(z) = 1\}$ can be obtained as a restriction of a smooth function on $\Phi_i(\overline{W}_i) \subset S^n$, hence **this map induces an isomorphism of the ring of smooth functions on $\overline{W}_i$ and its image.**

**Step 6:** Let $W_i$ be the set of interior points of $\overline{W}_i$. We have constructed a cover $\{W_i\}$ of $M$ such that on each $W_i$ the map $\Psi$ induced an isomorphism of ringed spaces. ■