# Geometry of manifolds

**lecture 3: partition of unity**

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## Sheaves of functions (reminder)

**DEFINITION: An open cover** of a topological space $X$ is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$.

**DEFINITION:** A **presheaf of functions** on a topological space $M$ is a collection of subrings $\mathcal{F}(U) \subset C(U)$ in the ring $C(U)$ of all functions on $U$, for each open subset $U \subset M$, such that the restriction of every $\gamma \in \mathcal{F}(U)$ to an open subset $U_1 \subset U$ belongs to $\mathcal{F}(U_1)$.

**DEFINITION:** A presheaf of functions $\mathcal{F}$ is called **a sheaf of functions** if these subrings satisfy the following condition. Let $\{U_i\}$ be a cover of an open subset $U \subset M$ (possibly infinite) and $f_i \in \mathcal{F}(U_i)$ a family of functions defined on the open sets of the cover and compatible on the pairwise intersections:

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that $f_i$ is the restriction of $f$ to $U_i$ for all $i$.**

**REMARK: A presheaf of functions** is a collection of subrings of functions on open subsets, compatible with restrictions. **A sheaf of fuctions is a presheaf allowing "gluing"** a function on a bigger open set if its restrictions to smaller open sets are compatible.

## Ringed spaces (reminder)

A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of functions. A **morphism** $(M, \mathcal{F}) \xrightarrow{\Psi} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \xrightarrow{\Psi} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $\psi^* f := f \circ \Psi$ belongs to the ring $\mathcal{F}\Big(\Psi^{-1}(U)\Big)$. An **isomorphism** of ringed spaces is a homeomorphism $\Psi$ such that $\Psi$ and $\Psi^{-1}$ are morphisms of ringed spaces.

**DEFINITION:** Let $(M, \mathcal{F})$ be a topological manifold equipped with a sheaf of functions. It is said to be a **smooth manifold of class** $C^\infty$ or $C^i$ if every point in $(M, \mathcal{F})$ has an open neighborhood isomorphic to the ringed space $(\mathbb{B}^n, \mathcal{F}')$, where $\mathbb{B}^n \subset \mathbb{R}^n$ is an open ball and $\mathcal{F}'$ is a ring of functions on an open ball $\mathbb{B}^n$ of this class.

**DEFINITION: Diffeomorphism** of smooth manifolds is a homeomorphism $\varphi$ which induces an isomorphims of ringed spaces, that is, $\varphi$ and $\varphi^{-1}$ map (locally defined) smooth functions to smooth functions.

**Assume from now on that all manifolds are Hausdorff and of class $C^\infty$.**

## Charts and coordinates (reminder)

**DEFINITION: Coordinate system** on a manifold $M$ is an open subset $V \subset M$ equipped with an isomorphism of ringed spaces $(V, C^\infty V) \cong (\mathbb{B}^n, C^\infty \mathbb{B}^n)$ per definition of a manifold.

**DEFINITION: A chart** on a smooth manifold $(M, C^\infty M)$ is an open subset $U \subset M$ together with an embedding $\psi: U \longrightarrow \mathbb{R}^n$ given by smooth functions $\varphi_1, ..., \varphi_n \in C^\infty M$ inducing a diffeomorphism on any open subset $V \subset U$ equipped with a coordinate system $(V, C^\infty V) \cong (\mathbb{B}^n, C^\infty \mathbb{B}^n)$.

**DEFINITION: Transition map** between two charts $\psi_1: U_1 \longrightarrow \mathbb{R}^n$ and $\psi_2: U_2 \longrightarrow \mathbb{R}^n$ is a map $\Psi_{ij}: \psi_1(U_1 \cap U_2) \longrightarrow \psi_2(U_1 \cap U_2)$ defined as $\Psi_{ij}(x) = \psi_2(\psi_1^{-1}(x))$.

**CLAIM: Transition maps are smooth.**

**Proof:** In local coordinates all functions $\varphi_1, ..., \varphi_n$ used in the definition of the transition map are smooth. ■

## Atlases on manifolds (reminder)

**DEFINITION: An atlas** on a smooth manifold is a collection of charts $\{U_i, \psi_i : U_i \longrightarrow \mathbb{R}^n\}$ satisfying $\bigcup U_i = M$ together with their transition maps.

**REMARK:** In such a situation, the **charts $U_i$ are usually identified with their images $\psi(U_i) \subset \mathbb{R}^n$.**

**REMARK: The sheaf $C^\infty M$ can be reconstructed from an atlas** as follows: a function $f$ on $U \subset M$ is smooth if and only if its restrictions to $U \cap U_i$ are smooth on all charts.

## Locally finite covers (reminder)

**DEFINITION:** An open cover $\{U_\alpha\}$ of a topological space $M$ is called **locally finite** if every point in $M$ has a neighborhood intersecting only a finite number of $U_\alpha$.

**CLAIM:** Let $\{U_\alpha\}$ be a locally finite atlas on a manifold $M$. **Then there exists a refinement $\{V_\beta\}$ of $\{U_\alpha\}$ such that a closure of each $V_\beta$ is compact in $M$.**

**THEOREM:** Let $\{U_\alpha\}$ be a countable locally finite cover of a Hausdorff topological space, such that a closure of each $U_\alpha$ is compact. **Then there exists another cover $\{V_\alpha\}$ indexed by the same set, such that $V_\alpha \Subset U_\alpha$.**

**REMARK:** If all $U_\alpha$ are diffeomorphic to $\mathbb{R}^n$, all $V_\alpha$ can be chosen diffeomorphic to an open ball. Indeed, any compact set is contained in an open ball.

**COROLLARY:** Let $M$ be a manifold admitting a locally finite countable cover $\{V_\alpha\}$, with $\varphi_\alpha : V_\alpha \longrightarrow \mathbb{R}^n$ diffeomorphisms. **Then there exists another atlas $\{U_\alpha, \varphi'_\alpha : U_\alpha \longrightarrow \mathbb{R}^n\}$, such that $\varphi'_\alpha(\mathbb{B})$ is also a cover of $M$, and $\mathbb{B} \subset \mathbb{R}^n$ a unit ball.** ■

## Construction of a partition of unity

**CLAIM: There exists a smooth function $\nu: \mathbb{R}^n \longrightarrow [0,1]$ which vanishes outside of a unit ball $\mathbb{B} \subset \mathbb{R}^n$ and is positive on $\mathbb{B}$.**

**Proof. Step 1:** There exists a **smooth function $a: \mathbb{R} \longrightarrow \mathbb{R}$ which is positive on $\mathbb{R}^{>0}$ and 0 on $\mathbb{R}^{\leqslant 0}$.** Take $a = e^{-x^{-2}}$ on $\mathbb{R}^{>0}$ and $a = 0$ on $\mathbb{R}^{\leqslant 0}$.

**Step 2:** There exists a **smooth function $b: \mathbb{R} \longrightarrow \mathbb{R}$ vanishing outside of $[0,1]$ and positive on the open interval $]0,1[$.** Take $b(x) = a(x)a(1-x)$.

**Step 3:** There exists a **smooth function $c: \mathbb{R} \longrightarrow \mathbb{R}$ equal to 0 on $]1,\infty[$ and equal to 1 on $]-\infty,0[$.** Take $c(x) = 1 - \lambda^{-1}\int_{-\infty}^{x} b(x)dx$, where $\lambda := \int_{-\infty}^{\infty} b(x)dx$.

**Step 4:** Now, let $\nu(z) := c(|z|^2)$. This function is smooth, vanishes on $|z| \geqslant 1$, and positive on $|z| < 1$. ■

**REMARK:** In assumptions of Corollary, let $\nu_\alpha(z) := \nu(\varphi'_\alpha)$, and $\mu_i := \frac{\nu_i}{\sum_\alpha \nu_\alpha}$. Then $\mu_\alpha: M \longrightarrow [0,1]$ are smooth functions with support in $U_\alpha$ satisfying $\sum_\alpha \mu_\alpha = 1$. Such a set of functions is called **a partition of unity**.

## Partition of unity: a formal definition

**DEFINITION:** Let $M$ be a smooth manifold and let $\{U_\alpha\}$ a locally finite cover of $M$. A **partition of unity** subordinate to the cover $\{U_\alpha\}$ is a family of smooth functions $f_i : M \to [0, 1]$ with compact support indexed by the same indices as the $U_i$'s and satisfying the following conditions.
(a) Every function $f_i$ vanishes outside $U_i$
(b) $\sum_i f_i = 1$

The argument of previous page proves the following theorem.

**THEOREM:** Let $\{U_\alpha\}$ be a countable, locally finite cover of a manifold $M$, with all $U_\alpha$ diffeomorphic to $\mathbb{R}^n$. **Then there exists a partition of unity subordinate to $\{U_\alpha\}$.** ■

## Whitney's theorem for compact manifolds

**THEOREM:** Let $M$ be a compact smooth manifold. **Then $M$ admits a closed smooth embedding to $\mathbb{R}^N$.**

**Proof. Step 1:** Choose a finite atlas $\{V_i, \varphi_i: V_i \longrightarrow \mathbb{R}^n, i = 1,2,...,m\}$, and subordinate partution of unity $\mu_i: M \longrightarrow [0,1]$. Let $\alpha: [0,1] \longrightarrow [0,1]$ be a smooth, monotonous function mapping 0 to 0 and $[1/2m, 1]$ to 1, and $\nu_i := \alpha(\mu_i)$.

**Step 2:** Denote by $W_i$ the set of interior points of $\overline{W}_i := \{z \mid \nu_i(z) = 1\} = \{z \mid \mu_i(z) \geqslant \frac{1}{2m}\}$. **Since $\sum_{i=1}^m \mu_i = 1$, the set $\{W_i\}$ is a cover of $M$.**

**Step 3:** For each $i$, the map $\Phi_i(z) := \frac{(\nu_i\varphi_i(z), 1-\nu_i(z))}{|(\nu_i\varphi_i(z), 1-\nu_i(z))|}$ **is smooth and induces a diffeomorphism of $W_i$ and an open subset of $S^n \subset \mathbb{R}^{n+1}$.**

**Step 4:** The product map

$$\Psi := \prod_{i=1}^m : \Phi_i : M \longrightarrow \underbrace{S^n \times S^n \times ... \times S^n}_{m \text{ times}}$$

is an injective, continuous map from a compact, hence **it is a homeomorphism to its image.** It is a smooth embedding, because its differential is injective. ■

## Embedding to $\mathbb{R}^\infty$

**QUESTION: What if $M$ is non-compact?**

**DEFINITION:** Define $\mathbb{R}^I_f$ as a direct sum of several copies of $\mathbb{R}$ indexed by a set $I$, that is, the set of points in a product where only finitely meny of coordinates can be non-zero. **The set $\mathbb{R}^I_f$ has metric**

$$d((x_1, ..., x_n, ...), (y_1, ..., y_n, ...)) := \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2 + ... + |x_n - y_n| + ...}$$

**It is well-defined, because only finitely many of $x_i, y_i$ are non-zero.**

**THEOREM:** Let $M$ be a compact smooth manifold, $\{V_i, \varphi_i : V_i \longrightarrow \mathbb{R}^n, i \in I\}$ be a locally finite atlas, and $\mu_i : M \longrightarrow [0,1]$ a subordinate partition of unity. Define $\nu_i := \alpha(\mu_i)$ and $\Phi_i$ as above, and let

$$\Psi := \prod_I : \Phi_i : M \longrightarrow \underbrace{S^n \times S^n \times ... \times S^n}_{I \text{ times}} \subset (\mathbb{R}^{n+1})^I$$

be the corresponding product map. Then **$\Psi$ is a homeomorphism to its image.**

## Embedding to $\mathbb{R}^\infty$ (cont.)

**THEOREM:** Let $M$ be a compact smooth manifold, $\{V_i, \varphi_i : V_i \longrightarrow \mathbb{R}^n, i \in I\}$ be a locally finite atlas, and $\mu_i : M \longrightarrow [0,1]$ a subordinate partition of unity. Define $\nu_i := \alpha(\mu_i)$ and $\Phi_i$ as above, and let

$$\Psi := \prod_I : \Phi_i : M \longrightarrow \underbrace{S^n \times S^n \times ... \times S^n}_{I \text{ times}} \subset (\mathbb{R}^{n+1})^I$$

be the corresponding product map. Then **$\Psi$ is a homeomorphism to its image.**

**Proof. Step 1:** $\Psi$ is injective by construction. To prove that it is a homeomorphism, it suffices to check that an image of an open set $U$ is open in $\Psi(M)$, for each $U \subset W_i$, for some open cover $\{W_i\}$

**Step 2:** However, the set $\Psi(W_i)$ is determined by $\nu_i(z) = 1$, that is, by $\Phi_i(z)_{n+1} = 1$, where $\Phi_i(z)_{n+1}$ is the last coordinate of $\Phi_i(z)$. Therefore, **$\Psi$ maps $W_i$ to an open subset of $\Psi(M)$.**

**Step 3:** Since $\Phi_i\big|_{\overline{W_i}}$ (restriction to a closure) is a continuous, bijective map from a compact, it's a homeomorphism. Therefore, **an image of any open subset $U \subset W_i$ is open in $\Psi(W_i)$, which is open in $\Psi(M)$ as follows from Step 2.** ■

## Paracompactness

**DEFINITION: An open cover** of a topological space $X$ is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$. A cover $\{V_i\}$ is a **refinement** of a cover $\{U_i\}$ if every $V_i$ is contained in some $U_i$.

**DEFINITION:** A cover of $M$ is called **locally finite** if any point $x \in M$ has a neighbourhood intersecting only finitely many of the elements of a cover.

**DEFINITION:** A topological space is called **paracompact** if any cover admits a locally finite refinement.

**EXERCISE:** Let $M$ be a paracompact topological space, and $Z \subset M$ a closed supset. **Prove that $Z$ is paracompact.**

## Paracompactness and partitions of unity

**THEOREM:** Let $M$ be a manifold. **Then the following conditions are equivalent:**

**(i)** $M$ **is metrizable**

**(ii)** $M$ **admits a partition of unity**

**(iii)** $M$ **is paracompact.**

**Proof:** Implication (iii) $\Rightarrow$ (ii) is proven above. Metrizability of $M$ follows from existence of partition of unity, because $M$ admits a homeomorphism to a subset of a metric space $\mathbb{R}^I$. This proves (ii) $\Rightarrow$ (i). It remains to prove that metrizability implies paracompactness.

We don't need it, but I will give a short sketch of a proof.

## Paracompactness and partitions of unity (cont.)

**Step 1:** Consider the function $\rho : M \longrightarrow \mathbb{R}^{\geqslant 0}$ mapping $x \in M$ to a supremum of all $r$ such that the open ball $B_r(x)$ is contained in one of $U_i$, and its closure is compact. It is easy to check that $\rho$ **is continuous,** and, indeed, 1-Lipschitz **(prove it).**

**Step 2:** Now we can replace $\{U_i\}$ by a cover $\{B_{\rho(x)}(x) \mid x \in M\}$, which is its refinement.

**Step 3:** Take a maximal subset $Z \subset M$ such that for each distinct $x, y \in Z$, one has $d(x,y) \geqslant 1/8\rho(x)$. Such a subset exists by Zorn's Lemma. Since $\rho$ is 1-Lipschitz, $\{W_i\} := \{B_{1/2\rho(x)}(x) \mid x \in Z\}$ is also a cover of $M$. It is a refinement of $\{U_i\}$, as follows from Step 2.

**Step 4:** Now, each $W_i = B_{1/2\rho(z_i)}(z_i)$ intersects only those $W_j = B_{1/2\rho(z_j)}(z_j)$ for which $d(z_i, z_j) \geqslant 1/8\rho(z_i)$, and there are only finitely many of them, by compactness of $\overline{W}_i$. ■

## Measure 0 subsets and Sard's theorem

**DEFINITION:** A subset $Z \subset \mathbb{R}^n$ has **measure zero** if, for every $\varepsilon > 0$, there exists a countable cover of $Z$ by open balls $U_i$ such that $\sum_i \operatorname{Vol} U_i < \varepsilon$.

**DEFINITION:** A subset $Z \subset M$ of a manifold $M$ has **measure 0** if intersection of $M$ with each chart $U_i \hookrightarrow \mathbb{R}^n$ has measure 0.

**Properties of measure 0 subsets**.

**A countable union of measure 0 subsets has measure 0.** A measure 0 subset $Z \subset M$ is **nowhere dense**, that is, $(M \backslash Z) \cap U \neq \emptyset$ for any non-empty open subset $U \subset M$.

**THEOREM: (a special case of Sard's Lemma)** Let $f: M \longrightarrow N$ be a smooth map of manifolds, $\dim M < \dim N$. **Then $f(M)$ has measure zero in $N$.**

**Its proof will be given in the next lecture** (if needed).

## Whitney's theorem (with a bound on dimension): strategy of the proof

**THEOREM:** Let $M$ be a smooth $n$-manifold. **Then $M$ admits a closed embedding to $\mathbb{R}^{2n+2}$.**

**Strategy of the proof:**

1. $M$ is embedded to $\mathbb{R}^\infty$.

2. We find a linear projection $\mathbb{R}^\infty \xrightarrow{\pi} \mathbb{R}^{2n+2}$ such that $\pi|_M$ is a closed embedding of manifolds.

**LEMMA:** Let $M \subset \mathbb{R}^I$ be a subset, and $\pi: \mathbb{R}^I \longrightarrow \mathbb{R}^J$ a linear projection. Consider the set $W$ of all vectors $\mathbb{R}(x-y)$, where $x, y \in M$ are distinct points. **Then $\pi|_M$ is injective if and only if $\ker \pi \cap W = 0$.**

**Proof:** $\pi|_M$ is not injective if and only if $\pi(x) = \pi(y)$, which is equivalent to $\pi(x-y) = 0$. ■

## Whitney's theorem: injectivity of projections

**REMARK:** Let $M \subset \mathbb{R}^I$ be a submanifold, and $W \subset \mathbb{R}^I$ the set of all vectors $\mathbb{R}(x-y)$, where $x, y \in M$ are distinct points. **Then $W$ is an image of a $2m+1$-dimensional manifold**, hence (by Sard's Lemma) **for any projection of $\mathbb{R}^I$ to a $(2m+2)$-dimensional space, image of $W$ has measure 0.**

**COROLLARY:** Let $M \subset \mathbb{R}^I$ be an $m$-dimensional submanifold, and $S \subset \mathbb{R}^I$ a maximal linear subspace not intersecting $W$. **Then the projection of $W$ to $\mathbb{R}^I/S$ is surjective.**

**Proof:** Suppose it's not surjective: $v \notin S$. Then $S \oplus \mathbb{R}v$ satisfies assumptions of lemma, hence $M \longrightarrow \mathbb{R}^I/(S + \mathbb{R}v)$ is also injective. ■

**THEOREM:** Let $M$ be a smooth $n$-manifold, $M \hookrightarrow \mathbb{R}^I$ an embedding constructed earlier. **Then there exists a projection $\pi : \mathbb{R}^I \longrightarrow \mathbb{R}^{2n+2}$ which is injective on $M$.**

**Proof:** Let $S$ be the maximal linear subspace such that the restriction of $\pi : \mathbb{R}^I \longrightarrow \mathbb{R}^I/S$ to $M$ is injective. Then the $2m+1$-dimensional manifold $W$ surjects to $\mathbb{R}^I/S$, hence $\dim \mathbb{R}^i/S \leqslant 2m+1$ by Sard's lemma. ■

## Tangent space to an embedded manifold

**DEFINITION:** Let $M \hookrightarrow \mathbb{R}^n$ be a smooth $m$-submanifold. The **tangent plane** at $p \in M$ is the plane in $\mathbb{R}^n$ tangent to $M$ (i.e, the plane lying in the image of the differential given in local coordinates). A **tangent vector** is an arbitrary vector in this plane with the origin at $p$. The space of all tangent vectors at $p$ is denoted by $T_pM$. Given a metric on $\mathbb{R}^n$, we can define the space of **unit tangent vectors** $\mathbb{S}^{m-1}M$ as the set of all pairs $(p, v)$, where $p \in M$, $v \in T_pM$, and $|v| = 1$.

**REMARK:** $\mathbb{S}^{m-1}M$ is a smooth manifold, projected to $M$ with fibers isomorphic to $m-1$-spheres, hence **$\mathbb{S}^{m-1}M$ is $(2m-1)$-dimensional.**

**LEMMA:** Let $M \subset \mathbb{R}^I$ be a subset, and $\pi: \mathbb{R}^I \longrightarrow \mathbb{R}^J$ a linear projection. Consider the set $W'$ of all vectors $\mathbb{R}t$, where $t \in T_xM$ **Then the differential $D\pi|_M$ is injective if and only if $\ker \pi \cap W' = 0$.** ■

Now the above argument is repeated: we take a maximal space $S \supset \mathbb{R}^I$ such that the restriction of $\pi: \mathbb{R}^I \longrightarrow \mathbb{R}^I/S$ to $M$ is injective and has injective differential, and the projection of $W \cup W'$ to $\mathbb{R}^I/S$ has to be surjective. However, $W'$ is an image of an $2m$-dimensional manifold $\mathbb{S}^{m-1}M \times \mathbb{R}$, hence **the projection of $W \cup W'$ to $\mathbb{R}^I/S$ can be surjective only if $\dim \mathbb{R}^I/S \leqslant 2m+2$.**

This proves Whitney's theorem.