Geometry of manifolds

Lecture 4: Hausdorff measure and Whitney's theorem

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Sheaves of functions (reminder)

DEFINITION: An open cover of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$.

DEFINITION: A **presheaf of functions** on a topological space M is a collection of subrings $\mathcal{F}(U) \subset C(U)$ in the ring $C(U)$ of all functions on U, for each open subset $U \subset M$, such that the restriction of every $\gamma \in \mathcal{F}(U)$ to an open subset $U_1 \subset U$ belongs to $\mathcal{F}(U_1)$.

DEFINITION: A presheaf of functions $\mathcal{F}$ is called **a sheaf of functions** if these subrings satisfy the following condition. Let $\{U_i\}$ be a cover of an open subset $U \subset M$ (possibly infinite) and $f_i \in \mathcal{F}(U_i)$ a family of functions defined on the open sets of the cover and compatible on the pairwise intersections:

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

REMARK: A presheaf of functions is a collection of subrings of functions on open subsets, compatible with restrictions. **A sheaf of fuctions is a presheaf allowing "gluing"** a function on a bigger open set if its restrictions to smaller open sets are compatible.

Ringed spaces (reminder)

A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of functions. A **morphism** $(M, \mathcal{F}) \xrightarrow{\Psi} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \xrightarrow{\Psi} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $\psi^* f := f \circ \Psi$ belongs to the ring $\mathcal{F}\big(\Psi^{-1}(U)\big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

DEFINITION: Let $(M, \mathcal{F})$ be a topological manifold equipped with a sheaf of functions. It is said to be a **smooth manifold of class** C^∞ or C^i if every point in $(M, \mathcal{F})$ has an open neighborhood isomorphic to the ringed space $(\mathbb{B}^n, \mathcal{F}')$, where $\mathbb{B}^n \subset \mathbb{R}^n$ is an open ball and $\mathcal{F}'$ is a ring of functions on an open ball $\mathbb{B}^n$ of this class.

DEFINITION: Diffeomorphism of smooth manifolds is a homeomorphism φ which induces an isomorphims of ringed spaces, that is, φ and φ^{-1} map (locally defined) smooth functions to smooth functions.

Assume from now on that all manifolds are Hausdorff and of class C^∞.

Partition of unity: a formal definition (reminder)

DEFINITION: Let M be a smooth manifold and let $\{U_\alpha\}$ a locally finite cover of M. A **partition of unity** subordinate to the cover $\{U_\alpha\}$ is a family of smooth functions $f_i : M \to [0,1]$ with compact support indexed by the same indices as the U_i's and satisfying the following conditions.
(a) Every function f_i vanishes outside U_i
(b) $\sum_i f_i = 1$

The argument of previous page proves the following theorem.

THEOREM: Let $\{U_\alpha\}$ be a countable, locally finite cover of a manifold M, with all U_α diffeomorphic to $\mathbb{R}^n$. **Then there exists a partition of unity subordinate to $\{U_\alpha\}$.** ■

Embedding to $\mathbb{R}^\infty$ (reminder)

DEFINITION: Define $\mathbb{R}^I_f$ as a direct sum of several copies of $\mathbb{R}$ indexed by a set I, that is, the set of points in a product where only finitely meny of coordinates can be non-zero. **The set $\mathbb{R}^I_f$ has metric**

$$d((x_1, ..., x_n, ...), (y_1, ..., y_n, ...)) := \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2 + ... + |x_n - y_n| + ...}$$

It is well-defined, because only finitely many of x_i, y_i are non-zero.

THEOREM: Let M be a compact smooth manifold, $\{V_i, \varphi_i : V_i \longrightarrow \mathbb{R}^n, i \in I\}$ be a locally finite atlas, and $\mu_i : M \longrightarrow [0,1]$ a subordinate partition of unity. Define $\nu_i := \alpha(\mu_i)$ and Φ_i as above, and let

$$\Psi := \prod_I : \Phi_i : M \longrightarrow \underbrace{S^n \times S^n \times ... \times S^n}_{I \text{ times}} \subset (\mathbb{R}^{n+1})^I$$

be the corresponding product map. Then **Ψ is a homeomorphism to its image.**

Borel measure

DEFINITION: Let C be a cube in $\mathbb{R}^n$ with edges parallel to coordinate axes of length r. Such a cube is called **normal**. Its **volume** is r^n.

DEFINITION: Let $S \subset \mathbb{R}^n$ be a closed subset. The **volume**, or **Borel measure** of S is an infimum of $\sum_i \operatorname{Vol}(S_i)$ for all (possibly, infinite) covers of S by normal cubes.

CLAIM: A subset $Z \subset \mathbb{R}^n$ has **measure zero** if for every $\varepsilon > 0$ there exists a countable cover of Z by cubes C_i such that $\sum_i \operatorname{Vol} C_i < \varepsilon$.

REMARK: Borel measure is a weaker form of Lebesgue measure, defined on closed subsets of $\mathbb{R}^n$, and equal to Lebesgue measure on those subsets.

Borel measure: axiomatic definition

THEOREM: (Properties of the volume)
Let $\mu(S)$ denote the measure of S. Then

(a) $\mu(\bigcup S_i) \leqslant \sum_i \mu(S_i)$.

(b) Measure is **monotonous:** for any $A \subset B$, $\mu(A) \leqslant \mu(B)$.

(c) Let $A = \bigcup A_i$ be an intersection of closed sets $A_0 \supset A_1 \supset ....$
Then $\mu(A) = \lim \mu(A_i)$.

(d) Measure is **additive**. Let $S = \bigcup_i S_i$ and $\mu(S_i \cap S_j) = 0$ for all $i \neq j$.
Then $\mu(S) = \sum_i \mu(S_i)$.

(e) Measure of a normal cube is l^n, where l is a length of its side.

Moreover, **for any closed set, its measure is determined uniquely by these properties.**

EXERCISE: Prove this theorem, using the following lemma.

LEMMA: Let $S \subset \mathbb{R}^n$ be a closed subset. Then $S = \bigcap S_i$, where $S_0 \supset S_1 \supset S_2 \supset ...$, and each S_i is a countable union of normal cubes, intersecting only in their faces.

Hausdorff measure and Hausdorff dimension

DEFINITION: Let M be a metric space. The **diameter** $\operatorname{diam} M \in [0, \infty]$ is the number $\sup_{x,y\in M} d(x,y)$.

DEFINITION: In a metric space, **a ball** $B_\varepsilon(x)$ of radius ε centered at x is defined as the set of all points y satisfying $d(x,y) < \varepsilon$.

DEFINITION: Let $\{S_i\}$ be a cover of a metric space M by balls of radius r with $r < \varepsilon$. Define $\mu_{d,\varepsilon} \in [0, \infty]$ as $\mu_{d,\varepsilon}(M) := \inf_{\{S_i\}} \sum_i (\operatorname{diam} S_i)^d$, where the infimum is taken over all such covers. The limit $\mu_d M := \sup \lim_{\varepsilon\to 0} \mu_{d,\varepsilon}(M)$ is called **d-dimensional Hausdorff measure** of M.

EXAMPLE: Let $M = \mathbb{R}^n$ with a metric d_∞ given by the norm $|(x_1, \dots, x_n)| := \max |x_i|$. **The balls in this metric are cubical**, and the (usual) volume of such a ball B is equal to $(\operatorname{diam} B)^n$. This gives $\mu_n(S) = \operatorname{Vol} S$ for each cube with sides parallel to coordinate planes.

COROLLARY: For $M = \mathbb{R}^n$ with the metric described above, **Hausdorff measure is equal to the Borel measure.**

Lipschitz maps

DEFINITION: A map $f : M \to N$ of metric spaces is called **Lipschitz with constant C** if $d(x,y) \geq C \cdot d\big(f(x), f(y)\big)$ for all $x, y \in M$. A map is called **bi-Lipschitz** if it is bijective and the inverse map is also Lipschitz.

EXAMPLE: A differentiable map $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is Lipschitz on each compact set B. Indeed, $d(x,y) \geqslant Cd(f(x), f(y))$, where $C = \sup_B |Df|$.

EXAMPLE: Let ν_1, ν_2 be norms on a vector space V, and d_1, d_2 the corresponding metrics. **The identity map $(V, d_1) \longrightarrow (V, d_2)$ is C-Lipschitz if and only if the unit ball $B_1(x, d_1)$ belongs to $B_C(x, d_2)$.**

CLAIM: Let $f : M \to N$ be a C-Lipschitz map. **Then the corresponding Hausdorff measures are related as $\mu_n(S) \geqslant C^n \mu_n(f(S))$.**

Proof: Let $\{S_i = B_{\varepsilon_i}(x_i)\}$ be a cover of S. Then $\{B_{C\varepsilon_i}(f(x_i))\}$ is a cover of $f(S)$. ■

COROLLARY:
Let $f : M \to N$ be a C-Lipschitz map. Then $\dim_H(M) \geqslant \dim_H(f(M))$. ■

Equivalent norms and Hausdorff measure

DEFINITION: Two norms on a vector space V are called **equivalent** if the identity map $(V, d_1) \longrightarrow (V, d_2)$ is bi-Lipschitz.

EXAMPLE: Since a unit cube in $\mathbb{R}^n$ contains a ball of radius 1, and is contained in a ball of radius $\sqrt{n}$, one has $|x|_{L^2} \geqslant |x|_{L^\infty} \geqslant \sqrt{n^{-1}}|x|_{L^2}$, where L^2 is the usual norm, and L^∞ the norm $|(x_1, \ldots, x_n)| := \max |x_i|$. Therefore, **the norms L^2 and L^∞ are equivalent**.

COROLLARY: Let $\mu_n^{L^2}$ denote the Hausdorff measure associated with the Euclidean metric on $\mathbb{R}^n$, and μ the usual (Borel) measure. **Then $\mu_n^{L^2}(S) \geqslant \mu(S) \geqslant \sqrt{n^{-n}}\mu_n^{L^2}(S)$.**

Proof: See the Claim above. ■

Hausdorff dimension

THEOREM: Let M be a metric space. Consider $\mu_d(M)$ as a function of d. **Then there exists a number $d_0 \in [0,\infty]$ such that $\mu_d(M) = \infty$ for $d < d_0$, and $\mu_d(M) = 0$ for $d > d_0$.**

Proof: Whenever $d' > d$, one has

$$\mu_{d',\varepsilon}(M) = \inf_{\{S_i\}} \sum_i (\operatorname{diam} S_i)^{d'} = \inf_{\{S_i\}} \sum_i (\operatorname{diam} S_i)^{d} (\operatorname{diam} S_i)^{d'-d} < \\ < \varepsilon^{d'-d} \inf_{\{S_i\}} \sum_i (\operatorname{diam} S_i)^{d} = \varepsilon^{d'-d} \mu_{d,\varepsilon}(M)$$

Passing to the limit $\varepsilon \to 0$, we obtain that $\mu_{d'}(M) \leqslant 0\mu_d(M)$. Therefore, $\mu_{d'}(M) = 0$ **whenever $\mu_d(M)$ is finite, and $\mu_d(M) = \infty$ whenever $\mu_{d'}(M) > 0$.** ■

DEFINITION: **Hausdorff dimension** $\dim_H(M)$ of a metric space is the number $\sup_{d \geqslant 0}\{\mu_d(M) = \infty\}$.

EXERCISE: Prove that set M has Hausdorff dimension 0 iff it is finite.

Lipschitz maps and Hausdorff dimension

CLAIM: Let f be a Lipschitz map. Then $\dim_H(f(M)) \leqslant \dim_H(M)$. ■

COROLLARY: A cube C in $\mathbb{R}^n$ has Hausdorff dimension n.

Proof: Indeed, C is bi-Lipschitz equivalent to the cube in L^∞-metric, but the Hausdorff measure associated with the L^∞-metric is equal to the usual volume. ■

CLAIM: Let $C = \bigcup C_i$ be a union of a countably many sets with $\mu_d C_i = 0$. **Then $\mu_d C = 0$.**

Proof: Take a cover $S_j(i)$ of C_i with $\sum_j (\operatorname{diam} S_j(i))^d < \delta^{i+1}$. Then $\{S_j(i)\}$ is a cover of C with $\sum_{j,i} (\operatorname{diam} S_j(i))^d < \delta$. ■

COROLLARY: $\mathbb{R}^n$ with the usual metric has Hausdorff dimension n.

Proof: Take a cover of $\mathbb{R}^n$ by coutably many unit cubes C_i. For $d > n$, $\mu_d(C_i) = 0$, hence $\mu_d(\mathbb{R}^n) = 0$. Since $\mu_n(C_i) > 0$, $\mu_n(\mathbb{R}^n)$ is also positive. ■

Hausdorff dimension of a manifold

THEOREM: Let $f: M \longrightarrow \mathbb{R}^n$ be a smooth map from a manifold, $\dim M < n$. **Suppose that M admits a countable cover by open balls with compact closure. Then $\mu_n f(M) = 0$.**

Proof: Let $B \subset \mathbb{R}^m$ be a closed ball, and $\varphi: B \longrightarrow \mathbb{R}^n$ a differentiable map. Then φ is Lipschitz, with the Lipschitz constant $C \leqslant \sup |D\varphi|$. The set M is covered by closed balls B_i, and $\mu_n(f(B_i)) = 0$, because $f|_{B_i}$ is Lipschitz, and $\dim_H B_i < n$. Using the Claim above, we obtain that $\mu_n(f(M)) = 0$. ■

DEFINITION: Hausdorff dimension of a subset $Z \subset M$ of a manifold is supremum of $\dim_h(Z \cup B)$ for all subsets $B \subset M$ equipped with a coordinate system.

COROLLARY: Let $f: M \longrightarrow N$ be a differentiable map of smooth manifolds, $\dim M < \dim N$. Suppose that M is covered by a countable number of open balls with compact closure. **Then $\mu_n(f(M)) = 0$.** ■

COROLLARY: (a version of Sard's lemma) Under these assumptions, $f(M)$ is nowhere dense.

Proof: Indeed, were it dense in an open ball B, one would have $\mu_n(f(M)) \geqslant \mu_n(B) > 0$, giving $\dim_H(f(M)) \geqslant n$, in contradiction to the corollary above. ■

Whitney's theorem (with a bound on dimension): strategy of the proof

THEOREM: Let M be a smooth n-manifold. **Then M admits a closed embedding to $\mathbb{R}^{2n+2}$.**

Strategy of the proof:

1. M is embedded to $\mathbb{R}^\infty$.

2. We find a linear projection $\mathbb{R}^\infty \xrightarrow{\pi} \mathbb{R}^{2n+2}$ such that $\pi|_M$ is a closed embedding of manifolds.

LEMMA: Let $M \subset \mathbb{R}^I$ be a subset, and $\pi : \mathbb{R}^I \longrightarrow \mathbb{R}^J$ a linear projection. Consider the set W of all vectors $\mathbb{R}(x-y)$, where $x, y \in M$ are distinct points. **Then $\pi|_M$ is injective if and only if $\ker \pi \cap W = 0$.**

Proof: $\pi|_M$ is not injective if and only if $\pi(x) = \pi(y)$, which is equivalent to $\pi(x-y) = 0$. ■

Whitney's theorem: injectivity of projections

REMARK: Let $M \subset \mathbb{R}^I$ be a submanifold, and $W \subset \mathbb{R}^I$ the set of all vectors $\mathbb{R}(x-y)$, where $x, y \in M$ are distinct points. **Then W is an image of a $2m+1$-dimensional manifold**, hence (by Sard's Lemma) **for any projection of $\mathbb{R}^I$ to a $(2m+2)$-dimensional space, image of W has measure 0.**

COROLLARY: Let $M \subset \mathbb{R}^I$ be an m-dimensional submanifold, and $S \subset \mathbb{R}^I$ a maximal linear subspace not intersecting W. **Then the projection of W to $\mathbb{R}^I/S$ is surjective.**

Proof: Suppose it's not surjective: $v \notin S$. Then $S \oplus \mathbb{R}v$ satisfies assumptions of lemma, hence $M \longrightarrow \mathbb{R}^I/(S+\mathbb{R}v)$ is also injective. ■

THEOREM: Let M be a smooth n-manifold, $M \hookrightarrow \mathbb{R}^I$ an embedding constructed earlier. **Then there exists a projection $\pi: \mathbb{R}^I \longrightarrow \mathbb{R}^{2n+2}$ which is injective on M.**

Proof: Let S be the maximal linear subspace such that the restriction of $\pi: \mathbb{R}^I \longrightarrow \mathbb{R}^I/S$ to M is injective. Then the $2m+1$-dimensional manifold W is mapped surjectively to $\mathbb{R}^I/S$, hence $\dim \mathbb{R}^i/S \leqslant 2m+1$ by Sard's lemma. ■

Tangent space to an embedded manifold

DEFINITION: Let $M \hookrightarrow \mathbb{R}^n$ be a smooth m-submanifold. The **tangent plane** at $p \in M$ is the plane in $\mathbb{R}^n$ tangent to M (i.e, the plane lying in the image of the differential given in local coordinates). A **tangent vector** is an arbitrary vector in this plane with the origin at p. The space of all tangent vectors at p is denoted by T_pM. Given a metric on $\mathbb{R}^n$, we can define the space of **unit tangent vectors** $\mathbb{S}^{m-1}M$ as the set of all pairs (p, v), where $p \in M$, $v \in T_pM$, and $|v| = 1$.

REMARK: $\mathbb{S}^{m-1}M$ is a smooth manifold, projected to M with fibers isomorphic to $m-1$-spheres, hence **$\mathbb{S}^{m-1}M$ is $(2m-1)$-dimensional.**

LEMMA: Let $M \subset \mathbb{R}^I$ be a subset, and $\pi: \mathbb{R}^I \longrightarrow \mathbb{R}^J$ a linear projection. Consider the set W' of all vectors $\mathbb{R}t$, where $t \in T_xM$ **Then the differential $D\pi|_M$ is injective if and only if $\ker \pi \cap W' = 0$.** ■

Now the above argument is repeated: we take a maximal space $S \supset \mathbb{R}^I$ such that the restriction of $\pi: \mathbb{R}^I \longrightarrow \mathbb{R}^I/S$ to M is injective and has injective differential, and the projection of $W \cup W'$ to $\mathbb{R}^I/S$ has to be surjective. However, W' is an image of an $2m$-dimensional manifold $\mathbb{S}^{m-1}M \times \mathbb{R}$, hence **the projection of $W \cup W'$ to $\mathbb{R}^I/S$ can be surjective only if $\dim \mathbb{R}^I/S \leqslant 2m+2$.**

This proves Whitney's theorem.