Geometry of manifolds

Lecture 4: Hausdorff measure and Whitney’s theorem

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Sheaves of functions (reminder)

**DEFINITION:** An open cover of a topological space $X$ is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$.

**DEFINITION:** A presheaf of functions on a topological space $M$ is a collection of subrings $\mathcal{F}(U) \subset C(U)$ in the ring $C(U)$ of all functions on $U$, for each open subset $U \subset M$, such that the restriction of every $\gamma \in \mathcal{F}(U)$ to an open subset $U_1 \subset U$ belongs to $\mathcal{F}(U_1)$.

**DEFINITION:** A presheaf of functions $\mathcal{F}$ is called a sheaf of functions if these subrings satisfy the following condition. Let $\{U_i\}$ be a cover of an open subset $U \subset M$ (possibly infinite) and $f_i \in \mathcal{F}(U_i)$ a family of functions defined on the open sets of the cover and compatible on the pairwise intersections:

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

for every pair of members of the cover. Then there exists $f \in \mathcal{F}(U)$ such that $f_i$ is the restriction of $f$ to $U_i$ for all $i$.

**REMARK:** A presheaf of functions is a collection of subrings of functions on open subsets, compatible with restrictions. A sheaf of functions is a presheaf allowing “gluing” a function on a bigger open set if its restrictions to smaller open sets are compatible.
Ringed spaces (reminder)

A **ringed space** \((M, \mathcal{F})\) is a topological space equipped with a sheaf of functions. A **morphism** \(\Psi: (M, \mathcal{F}) \to (N, \mathcal{F}')\) of ringed spaces is a continuous map \(M \to N\) such that, for every open subset \(U \subset N\) and every function \(f \in \mathcal{F}'(U)\), the function \(\psi^* f := f \circ \Psi\) belongs to the ring \(\mathcal{F}(\Psi^{-1}(U))\). An **isomorphism** of ringed spaces is a homeomorphism \(\Psi\) such that \(\Psi\) and \(\Psi^{-1}\) are morphisms of ringed spaces.

**DEFINITION:** Let \((M, \mathcal{F})\) be a topological manifold equipped with a sheaf of functions. It is said to be a **smooth manifold of class** \(C^\infty\) or \(C^i\) if every point in \((M, \mathcal{F})\) has an open neighborhood isomorphic to the ringed space \((\mathbb{B}^n, \mathcal{F}')\), where \(\mathbb{B}^n \subset \mathbb{R}^n\) is an open ball and \(\mathcal{F}'\) is a ring of functions on an open ball \(\mathbb{B}^n\) of this class.

**DEFINITION:** **Diffeomorphism** of smooth manifolds is a homeomorphism \(\varphi\) which induces an isomorphisms of ringed spaces, that is, \(\varphi\) and \(\varphi^{-1}\) map (locally defined) smooth functions to smooth functions.

**Assume from now on that all manifolds are Hausdorff and of class** \(C^\infty\).
Partition of unity: a formal definition (reminder)

**DEFINITION:** Let $M$ be a smooth manifold and let $\{U_\alpha\}$ a locally finite cover of $M$. A partition of unity subordinate to the cover $\{U_\alpha\}$ is a family of smooth functions $f_i : M \to [0,1]$ with compact support indexed by the same indices as the $U_i$’s and satisfying the following conditions.

(a) Every function $f_i$ vanishes outside $U_i$
(b) $\sum_i f_i = 1$

The argument of previous page proves the following theorem.

**THEOREM:** Let $\{U_\alpha\}$ be a countable, locally finite cover of a manifold $M$, with all $U_\alpha$ diffeomorphic to $\mathbb{R}^n$. Then there exists a partition of unity subordinate to $\{U_\alpha\}$. ■
Embedding to $\mathbb{R}^\infty$ (reminder)

**DEFINITION:** Define $\mathbb{R}^I_f$ as a direct sum of several copies of $\mathbb{R}$ indexed by a set $I$, that is, the set of points in a product where only finitely many of coordinates can be non-zero. **The set $\mathbb{R}^I_f$ has metric**

$$d((x_1, \ldots, x_n, \ldots), (y_1, \ldots, y_n, \ldots)) := \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2 + \ldots + |x_n - y_n| + \ldots}.$$ 

It is well-defined, because only finitely many of $x_i, y_i$ are non-zero.

**THEOREM:** Let $M$ be a compact smooth manifold, $\{V_i, \varphi_i : V_i \rightarrow \mathbb{R}^n, i \in I\}$ be a locally finite atlas, and $\mu_i : M \rightarrow [0,1]$ a subordinate partition of unity. Define $\nu_i := \alpha(\mu_i)$ and $\Phi_i$ as above, and let

$$\Psi := \prod_I: \Phi_i : M \rightarrow S^n \times S^n \times \ldots \times S^n \subset (\mathbb{R}^{n+1})^I$$

be the corresponding product map. Then $\Psi$ is a homeomorphism to its image.
Borel measure

**DEFINITION:** Let $C$ be a cube in $\mathbb{R}^n$ with edges parallel to coordinate axes of length $r$. Such a cube is called normal. Its volume is $r^n$.

**DEFINITION:** Let $S \subset \mathbb{R}^n$ be a closed subset. The volume, or Borel measure of $S$ is an infimum of $\sum_i \text{Vol}(S_i)$ for all (possibly, infinite) covers of $S$ by normal cubes.

**CLAIM:** A subset $Z \subset \mathbb{R}^n$ has measure zero if for every $\varepsilon > 0$ there exists a countable cover of $Z$ by cubes $C_i$ such that $\sum_i \text{Vol} C_i < \varepsilon$.

**REMARK:** Borel measure is a weaker form of Lebesgue measure, defined on closed subsets of $\mathbb{R}^n$, and equal to Lebesgue measure on those subsets.
Borel measure: axiomatic definition

**THEOREM: (Properties of the volume)**
Let $\mu(S)$ denote the measure of $S$. Then

(a) $\mu(\bigcup S_i) \leq \sum_i \mu(S_i)$.

(b) Measure is **monotonous**: for any $A \subset B$, $\mu(A) \leq \mu(B)$.

(c) Let $A = \bigcup A_i$ be an intersection of closed sets $A_0 \supset A_1 \supset \ldots$. Then $\mu(A) = \lim \mu(A_i)$.

(d) Measure is **additive**. Let $S = \bigcup_i S_i$ and $\mu(S_i \cap S_j) = 0$ for all $i \neq j$. Then $\mu(S) = \sum_i \mu(S_i)$.

(e) Measure of a normal cube is $l^n$, where $l$ is a length of its side.

Moreover, for any closed set, its measure is determined uniquely by these properties.

**EXERCISE:** Prove this theorem, using the following lemma.

**LEMMA:** Let $S \subset \mathbb{R}^n$ be a closed subset. Then $S = \bigcap S_i$, where $S_0 \supset S_1 \supset S_2 \supset \ldots$, and each $S_i$ is a countable union of normal cubes, intersecting only in their faces.
Hausdorff measure and Hausdorff dimension

**DEFINITION:** Let $M$ be a metric space. The **diameter** $\text{diam} M \in [0, \infty]$ is the number $\sup_{x,y \in M} d(x, y)$.

**DEFINITION:** In a metric space, a **ball** $B_\varepsilon(x)$ of radius $\varepsilon$ centered at $x$ is defined as the set of all points $y$ satisfying $d(x, y) < \varepsilon$.

**DEFINITION:** Let $\{S_i\}$ be a cover of a metric space $M$ by balls of radius $r$ with $r < \varepsilon$. Define $\mu_{d,\varepsilon} \in [0, \infty]$ as $\mu_{d,\varepsilon}(M) := \inf \{ \sum_i (\text{diam} S_i)^d \}$, where the infimum is taken over all such covers. The limit $\mu_d M := \sup \lim_{\varepsilon \to 0} \mu_{d,\varepsilon}(M)$ is called the **$d$-dimensional Hausdorff measure** of $M$.

**EXAMPLE:** Let $M = \mathbb{R}^n$ with a metric $d_\infty$ given by the norm $|(x_1, \ldots, x_n)| := \max |x_i|$. The balls in this metric are cubical, and the (usual) volume of such a ball $B$ is equal to $(\text{diam} B)^n$. This gives $\mu_n(S) = \text{Vol} S$ for each cube with sides parallel to coordinate planes.

**COROLLARY:** For $M = \mathbb{R}^n$ with the metric described above, Hausdorff measure is equal to the Borel measure.
Lipschitz maps

**DEFINITION:** A map \( f : M \to N \) of metric spaces is called **Lipschitz with constant** \( C \) if \( d(x, y) \geq C \cdot d(f(x), f(y)) \) for all \( x, y \in M \). A map is called **bi-Lipschitz** if it is bijective and the inverse map is also Lipschitz.

**EXAMPLE:** A differentiable map \( f : \mathbb{R}^n \to \mathbb{R}^m \) is Lipschitz on each compact set \( B \). Indeed, \( d(x, y) \geq C d(f(x), f(y)) \), where \( C = \sup_B |Df| \).

**EXAMPLE:** Let \( \nu_1, \nu_2 \) be norms on a vector space \( V \), and \( d_1, d_2 \) the corresponding metrics. The identity map \( (V, d_1) \to (V, d_2) \) is \( C \)-Lipschitz if and only if the unit ball \( B_1(x, d_1) \) belongs to \( B_C(x, d_2) \).

**CLAIM:** Let \( f : M \to N \) be a \( C \)-Lipschitz map. Then the corresponding Hausdorff measures are related as \( \mu_n(S) \geq C^n \mu_n(f(S)) \).

**Proof:** Let \( \{S_i = B_{\varepsilon_i}(x_i)\} \) be a cover of \( S \). Then \( \{B_{C\varepsilon_i}(f(x_i))\} \) is a cover of \( f(S) \). ■

**COROLLARY:** Let \( f : M \to N \) be a \( C \)-Lipschitz map. Then \( \dim_H(M) \geq \dim_H(f(M)) \). ■
Equivalent norms and Hausdorff measure

**DEFINITION:** Two norms on a vector space $V$ are called equivalent if the identity map $(V,d_1) \to (V,d_2)$ is bi-Lipschitz.

**EXAMPLE:** Since a unit cube in $\mathbb{R}^n$ contains a ball of radius 1, and is contained in a ball of radius $\sqrt{n}$, one has $|x|_{L^2} \geq |x|_{L^\infty} \geq \sqrt{n-1}|x|_{L^2}$, where $L^2$ is the usual norm, and $L^\infty$ the norm $|(x_1,\ldots,x_n)| := \max |x_i|$. Therefore, the norms $L^2$ and $L^\infty$ are equivalent.

**COROLLARY:** Let $\mu_n^{L^2}$ denote the Hausdorff measure associated with the Euclidean metric on $\mathbb{R}^n$, and $\mu$ the usual (Borel) measure. Then $\mu_n^{L^2}(S) \geq \mu(S) \geq \sqrt{n-n} \mu_n^{L^2}(S)$.

**Proof:** See the Claim above. ■
**Hausdorff dimension**

**THEOREM:** Let $M$ be a metric space. Consider $\mu_d(M)$ as a function of $d$. Then there exists a number $d_0 \in [0, \infty]$ such that $\mu_d(M) = \infty$ for $d < d_0$, and $\mu_d(M) = 0$ for $d > d_0$.

**Proof:** Whenever $d' > d$, one has

$$
\mu_{d',\varepsilon}(M) = \inf_{\{S_i\}} \sum_i (\text{diam}S_i)^{d'} = \inf_{\{S_i\}} \sum_i (\text{diam}S_i)^d(\text{diam}S_i)^{d'-d} <
$$

$$
< \varepsilon^{d'-d} \inf_{\{S_i\}} \sum_i (\text{diam}S_i)^d = \varepsilon^{d'-d} \mu_{d,\varepsilon}(M)
$$

Passing to the limit $\varepsilon \to 0$, we obtain that $\mu_{d'}(M) \leq 0 \mu_d(M)$. Therefore, $\mu_{d'}(M) = 0$ whenever $\mu_d(M)$ is finite, and $\mu_d(M) = \infty$ whenever $\mu_{d'}(M) > 0$. ■

**DEFINITION:** **Hausdorff dimension** $\dim_H(M)$ of a metric space is the number $\sup_{d \geq 0} \{\mu_d(M) = \infty\}$.

**EXERCISE:** Prove that set $M$ has Hausdorff dimension 0 iff it is finite.
Lipschitz maps and Hausdorff dimension

CLAIM: Let \( f \) be a Lipschitz map. Then \( \dim_H(f(M)) \leq \dim_H(M) \). ■

COROLLARY: A cube \( C \) in \( \mathbb{R}^n \) has Hausdorff dimension \( n \).

Proof: Indeed, \( C \) is bi-Lipschitz equivalent to the cube in \( L^\infty \)-metric, but
the Hausdorff measure associated with the \( L^\infty \)-metric is equal to the usual
volume. ■

CLAIM: Let \( C = \bigcup C_i \) be a union of a countably many sets with \( \mu_d C_i = 0 \).
Then \( \mu_d C = 0 \).

Proof: Take a cover \( S_j(i) \) of \( C_i \) with \( \sum_j (\text{diam} S_j(i))^d < \delta^{i+1} \). Then \( \{ S_j(i) \} \) is
a cover of \( C \) with \( \sum_{j,i} (\text{diam} S_j(i))^d < \delta \). ■

COROLLARY: \( \mathbb{R}^n \) with the usual metric has Hausdorff dimension \( n \).

Proof: Take a cover of \( \mathbb{R}^n \) by countably many unit cubes \( C_i \). For \( d > n \),
\( \mu_d(C_i) = 0 \), hence \( \mu_d(\mathbb{R}^n) = 0 \). Since \( \mu_n(C_i) > 0 \), \( \mu_n(\mathbb{R}^n) \) is also positive. ■
Hausdorff dimension of a manifold

**THEOREM:** Let \( f : M \rightarrow \mathbb{R}^n \) be a smooth map from a manifold, \( \dim M < n \). Suppose that \( M \) admits a countable cover by open balls with compact closure. Then \( \mu_n f(M) = 0 \).

**Proof:** Let \( B \subset \mathbb{R}^m \) be a closed ball, and \( \varphi : B \rightarrow \mathbb{R}^n \) a differentiable map. Then \( \varphi \) is Lipschitz, with the Lipschitz constant \( C \leq \sup |D\varphi| \). The set \( M \) is covered by closed balls \( B_i \), and \( \mu_n(f(B_i)) = 0 \), because \( f|_{B_i} \) is Lipschitz, and \( \dim_H B_i < n \). Using the Claim above, we obtain that \( \mu_n(f(M)) = 0 \).

**DEFINITION:** Hausdorff dimension of a subset \( Z \subset M \) of a manifold is supremum of \( \dim_h(Z \cup B) \) for all subsets \( B \subset M \) equipped with a coordinate system.

**COROLLARY:** Let \( f : M \rightarrow N \) be a differentiable map of smooth manifolds, \( \dim M < \dim N \). Suppose that \( M \) is covered by a countable number of open balls with compact closure. Then \( \mu_n(f(M)) = 0 \).

**COROLLARY:** (a version of Sard’s lemma) Under these assumptions, \( f(M) \) is nowhere dense.

**Proof:** Indeed, were it dense in an open ball \( B \), one would have \( \mu_n(f(M)) \geq \mu_n(B) > 0 \), giving \( \dim_H(f(M)) \geq n \), in contradiction to the corollary above.
Whitney's theorem (with a bound on dimension): strategy of the proof

**THEOREM:** Let $M$ be a smooth $n$-manifold. Then $M$ admits a closed embedding to $\mathbb{R}^{2n+2}$.

**Strategy of the proof:**
1. $M$ is embedded to $\mathbb{R}^\infty$.
2. We find a linear projection $\mathbb{R}^\infty \xrightarrow{\pi} \mathbb{R}^{2n+2}$ such that $\pi|_M$ is a closed embedding of manifolds.

**LEMMA:** Let $M \subset \mathbb{R}^I$ be a subset, and $\pi : \mathbb{R}^I \to \mathbb{R}^J$ a linear projection. Consider the set $W$ of all vectors $\mathbb{R}(x - y)$, where $x, y \in M$ are distinct points. Then $\pi|_M$ is injective if and only if $\ker \pi \cap W = 0$.

**Proof:** $\pi|_M$ is not injective if and only if $\pi(x) = \pi(y)$, which is equivalent to $\pi(x - y) = 0$. □
Whitney's theorem: injectivity of projections

**REMARK:** Let \( M \subset \mathbb{R}^I \) be a submanifold, and \( W \subset \mathbb{R}^I \) the set of all vectors \( \mathbb{R}(x-y) \), where \( x, y \in M \) are distinct points. Then \( W \) is an image of a \( 2m+1 \)-dimensional manifold, hence (by Sard's Lemma) for any projection of \( \mathbb{R}^I \) to a \((2m+2)\)-dimensional space, image of \( W \) has measure 0.

**COROLLARY:** Let \( M \subset \mathbb{R}^I \) be an \( m \)-dimensional submanifold, and \( S \subset \mathbb{R}^I \) a maximal linear subspace not intersecting \( W \). Then the projection of \( W \) to \( \mathbb{R}^I/S \) is surjective.

**Proof:** Suppose it's not surjective: \( v \notin S \). Then \( S \oplus \mathbb{R}v \) satisfies assumptions of lemma, hence \( M \rightarrow \mathbb{R}^I/(S + \mathbb{R}v) \) is also injective. ■

**THEOREM:** Let \( M \) be a smooth \( n \)-manifold, \( M \hookrightarrow \mathbb{R}^I \) an embedding constructed earlier. Then there exists a projection \( \pi : \mathbb{R}^I \rightarrow \mathbb{R}^{2n+2} \) which is injective on \( M \).

**Proof:** Let \( S \) be the maximal linear subspace such that the restriction of \( \pi : \mathbb{R}^I \rightarrow \mathbb{R}^I/S \) to \( M \) is injective. Then the \( 2m+1 \)-dimensional manifold \( W \) is mapped surjectively to \( \mathbb{R}^I/S \), hence \( \dim \mathbb{R}^i/S \leq 2m+1 \) by Sard's lemma. ■
Tangent space to an embedded manifold

**DEFINITION:** Let $M \hookrightarrow \mathbb{R}^n$ be a smooth $m$-submanifold. The **tangent plane** at $p \in M$ is the plane in $\mathbb{R}^n$ tangent to $M$ (i.e., the plane lying in the image of the differential given in local coordinates). A **tangent vector** is an arbitrary vector in this plane with the origin at $p$. The space of all tangent vectors at $p$ is denoted by $T_p M$. Given a metric on $\mathbb{R}^n$, we can define the space of **unit tangent vectors** $S^{m-1} M$ as the set of all pairs $(p, v)$, where $p \in M$, $v \in T_p M$, and $|v| = 1$.

**REMARK:** $S^{m-1} M$ is a smooth manifold, projected to $M$ with fibers isomorphic to $m-1$-spheres, hence $S^{m-1} M$ is $(2m - 1)$-dimensional.

**LEMMA:** Let $M \subset \mathbb{R}^I$ be a subset, and $\pi : \mathbb{R}^I \rightarrow \mathbb{R}^J$ a linear projection. Consider the set $W'$ of all vectors $\mathbb{R} t$, where $t \in T_x M$ Then the differential $D\pi|_M$ is injective if and only if $\ker \pi \cap W' = 0$. ■

Now the above argument is repeated: we take a maximal space $S \supset \mathbb{R}^I$ such that the restriction of $\pi : \mathbb{R}^I \rightarrow \mathbb{R}^I/S$ to $M$ is injective and has injective differential, and the projection of $W \cup W'$ to $\mathbb{R}^I/S$ has to be surjective. However, $W'$ is an image of an $2m$-dimensional manifold $S^{m-1} M \times \mathbb{R}$, hence the projection of $W \cup W'$ to $\mathbb{R}^I/S$ can be surjective only if $\dim \mathbb{R}^I/S \leq 2m + 2$.

This proves Whitney’s theorem.