Geometry of manifolds

Lecture 5: Derivations in rings

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Rings and derivations

REMARK: All rings in these handouts are assumed to be commutative and with unit. Algebras are associative, but not necessarily commutative (such as the matrix algebra). **Rings over a field k** are rings containing a field k. We assume that k has characteristic 0.

DEFINITION: Let R be a ring over a field k. A k-linear map $D\ R \longrightarrow R$ is called **a derivation** if it satisfies **the Leibnitz equation** $D(fg) = D(f)g + gD(f)$. The space of derivations is denoted as $\operatorname{Der}_k(R)$.

EXAMPLE: $\frac{d}{dt}: \mathbb{C}[t] \longrightarrow \mathbb{C}[t]$. $\frac{d}{dt}: C^\infty\mathbb{R} \longrightarrow C^\infty\mathbb{R}$.

REMARK: Any derivation $\delta \in \operatorname{Der}_k(R)$ vanishes on $k \subset R$. Indeed, $\delta(1) = \delta(1 \cdot 1) = 2\delta(1)$.

CLAIM: Let K be **a finite extension** of a field k, that is, a field containing k and finite-dimensional as a K-linear space. **Then $\operatorname{Der}_k(K) = 0$.**

Proof: Indeed, any $x \in K$ satisfies a non-trivial polynomial equation $P(x) = 0$ with coefficients in k. Chose $P(t)$ of smallest degree possible. **For any $\delta \in \operatorname{Der}_k(R)$, we have $0 = \delta(P(x)) = P'(x)\delta(x)$,** and unless $\delta(x) = 0$, one has $P'(x) = 0$, giving a contradiction. ∎

Modules over a ring

DEFINITION: Let R be a ring over a field k. **An R-module** is a vector space V over k, equipped with an algebra homomorphism $R \longrightarrow \operatorname{End}(V)$, where $\operatorname{End}(V)$ denotes the endomorphism algebra of V, that is, the matrix algebra.

REMARK: Let R be a field. Then R-modules are the same as vector spaces over R.

DEFINITION: Homomorphisms, isomorphisms, submodules, quotient modules, direct sums of modules are defined in the same way as for the vector spaces. A ring R is itself an R-module. A direct sum of n copies of R is denoted R^n. Such R-module is called **a free R-module**.

EXAMPLE: **R-submodules in R are the same as ideals in R.**

DEFINITION: **Finitely generated** R-module is a quotient module of R^n.

Noetherian rings

DEFINITION: A Noetherian ring is a ring R with all ideals finitely generated as R-modules.

THEOREM: Let R be a Noetherian ring. **Then any submodule of a finitely generated R-module is finitely generated.**

Proof. Step 1: Consider an exact sequence of R-modules $0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$. Then M is called **an extension** of M_1 and M_2. An extension of finitely-generated modules is finitely generated. Indeed, take a finite set of generators in M_2, and let $\{\xi_i\}$ be preimages of these generators in M. Let $\{\zeta_j\}$ be a finite set of generators in $M_1 \subset M$. **Then $\{\zeta_j + \xi_i\}$ generate M.**

Step 2: A filtration on a module M as a sequence of submodules $0 = M_0 \subset M_1 \subset ... \subset M_n = M$. From Step 1 and induction it follows that any **M admitting a filtration with finitely-generated M_i/M_{i-1} is also finitely-generated.**

Step 3: Let $M \subset R^n = W$. Consider a filtration $W_0 \subset W_1 \subset ... \subset W_n = W$, with $W_i = R^i$, and let $M_i = M \cap W_i$. **Then M_i/M_{i-1} is a submodule of $W_i/W_{i-1} = R$, hence finitely-generated.** ■

Ring of smooth functions

THEOREM: Let R be a ring of smooth real functions on $\mathbb{R}^n$. **Then R is not Noetherian.** Moreover, the ideal I of all functions with all derivatives vanishing at 0 is not finitely generated.

Proof. Step 1: If I is generated by $f_1, ..., f_n$, then for each $g \in I$, one can express g as $g = \sum g_i f_i$. **Then**

$$\limsup_{x \to 0} \frac{g}{\sum f_i^2} \leqslant \sum |g_i| \frac{f_1}{\sum f_i^2} < \infty.$$

Step 2: Let x_i be coordinate functions. **The function $F := \frac{\sum f_i^2}{\sum x_i^2}$ is smooth**, and all its derivatives vanish at 0, however,

$$\limsup_{x \to 0} \frac{F}{\sum f_i^2} = \limsup_{x \to 0} \frac{1}{\sum x_i^2} = \infty.$$

■

Derivations as an R-module

REMARK: Let R be a ring over k. **The space $\operatorname{Der}_k(R)$ of derivations is also an R-module,** with multiplicative action of R given by $rD(f) = rD(f)$.

CLAIM: Let $R = k[t_1, .., t_k]$ be a polynomial ring. **Then $\operatorname{Der}_k(R)$ is a free R-module isomorphic to R^n,** with generators $\frac{d}{dt_1}, \frac{d}{dt_2}, ..., \frac{d}{dt_n}$.

Proof: Consider a map $\operatorname{Der}_k(R) \longrightarrow R^n$,

$$D \longrightarrow (D(t_1), D(t_2), ..., D(t_n))$$

It is surjective, because it maps each $\frac{d}{dt_i}$ to $(0, ..., 0, 1, 0, ..., 0)$, and injective, because each derivation which vanishes on t_i, vanishes on the whole polynomial ring. ■

Now we prove a similar result for $C^\infty \mathbb{R}^n$.

Hadamard's Lemma

LEMMA: (Hadamard's Lemma)
Let f be a smooth function f on $\mathbb{R}^n$, and x_i the coordinate functions. **Then $f(x) = f(0) + \sum_{i=1}^n x_i g_i(x)$, for some smooth $g_i \in C^\infty \mathbb{R}^n$.**

Proof: Let $t \in \mathbb{R}^n$. Consider a function $h(t) \in C^\infty \mathbb{R}^n$, $h(t) = f(tx)$. Then $\frac{dh}{dt} = \sum \frac{df(tx)}{dx_i}(tx) x_i$, giving

$$f(x) - f(0) = \int_0^1 \frac{dh}{dt} dt = \sum_i x_i \int_0^1 \frac{df(tx)}{dx_i}(tx) dt.$$

■

COROLLARY: Let $\mathfrak{m}_0$ be an ideal of all smooth functions on $\mathbb{R}^n$ vanishing in 0. **Then $\mathfrak{m}_0$ is generated by coordinate functions.** ■

COROLLARY: Let f be a smooth function on $\mathbb{R}^n$ satisfying $f(x) = 0$ and $f'(x) = 0$. **Then $f \in \mathfrak{m}_x^2$.**

Proof: $f(x) = \sum_{i=1}^n x_i g_i(x)$, where all g_i vanish in 0. ■

Derivations of $C^\infty \mathbb{R}^n$

THEOREM: Let $x_1, ..., x_n$ be coordinates on $\mathbb{R}^n$, $R = C^\infty \mathbb{R}^n$, and $\operatorname{Der}(R) \stackrel{\Pi}{\longrightarrow} (C^\infty \mathbb{R}^n)^n$ map D to $(D(x_1), D(x_2), ..., D(x_n))$. **Then $D: \operatorname{Der}(C^\infty \mathbb{R}^n) \longrightarrow R^n$ is an isomorphism.**

Proof. Step 1: Since Π maps each $\frac{d}{dt_i}$ to $(0, ..., 0, 1, 0, ..., 0)$, it is **surjective.**

Step 2: Let $\mathfrak{m}_0$ be an ideal of 0, and $D \subset \ker \Pi$. Then $\Pi(x_i) = 0$, where x_i are coordinate functions. By Hadamard's Lemma, $f(x) = f(0) + \sum_{i=1}^n x_i g_i(x)$, hence $D(f) = \sum_{i=1}^n x_i D(g_i)$. **Therefore, $D(f)$ lies in $\mathfrak{m}_0$.**

Step 3: Same argument proves that $D(f)$ vanishes everywhere, for all $f \in C^\infty M$. ■

Sheaves

DEFINITION: An open cover of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$.

REMARK: The definition of a sheaf below is a more abstract version of the notion of "sheaf of functions" defined previously.

DEFINITION: A **presheaf** on a topological space M is a collection of vector spaces $\mathcal{F}(U)$, for each open subset $U \subset M$, together with **restriction maps** $R_{UW}\mathcal{F}(U) \longrightarrow \mathcal{F}(W)$ defined for each $W \subset U$, such that for any three open sets $W \subset V \subset U$, $\Psi_{UW} = \Psi_{UV} \circ \Psi_{VW}$. Elements of $\mathcal{F}(U)$ are called **sections of $\mathcal{F}$ over U**, and restriction map often denoted $f|_W$

DEFINITION: A presheaf $\mathcal{F}$ is called **a sheaf** if for any open set U and any cover $U = \bigcup U_I$ the following two conditions are satisfied.

1. Let $f \in \mathcal{F}(U)$ be a section of $\mathcal{F}$ on U such that its restriction to each U_i vanishes. **Then $f = 0$.**

2. Let $f_i \in \mathcal{F}(U_i)$ be a family of sections compatible on the pairwise intersections: $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

Sheaves and exact sequences

DEFINITION: A sequence $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow ...$ of homomorphisms of abelian groups or vector spaces is called **exact** if the image of each map is the kernel of the next one.

CLAIM: A presheaf $\mathcal{F}$ is a sheaf if and only if for every cover $\{U_i\}$ of an open subset $U \subset M$, **the sequence of restriction maps**

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$$

is exact, with $\eta \in \mathcal{F}(U_i)$ mapped to $\eta\big|_{U_i \cap U_j}$ and $-\eta\big|_{U_j \cap U_i}$.

Ringed spaces (reminder)

DEFINITION: A sheaf of rings is a sheaf $\mathcal{F}$ such that all the spaces $\mathcal{F}(U)$ are rings, and all restriction maps are ring homomorphisms.

DEFINITION: A sheaf of functions is a subsheaf in a sheaf of all functions, closed under multiplication.

For simplicity, I assume now that a sheaf of rings is a subsheaf in a sheaf of all functions.

DEFINITION: A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of rings. A **morphism** $(M, \mathcal{F}) \xrightarrow{\Psi} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \xrightarrow{\Psi} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $\psi^* f := f \circ \Psi$ belongs to the ring $\mathcal{F}\Big(\Psi^{-1}(U)\Big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

Smooth manifold (reminder)

DEFINITION: Let $(M, \mathcal{F})$ be a topological manifold equipped with a sheaf of functions. It is said to be a **smooth manifold of class** C^∞ or C^i if every point in $(M, \mathcal{F})$ has an open neighborhood isomorphic to the ringed space $(\mathbb{B}^n, \mathcal{F}')$, where $\mathbb{B}^n \subset \mathbb{R}^n$ is an open ball and $\mathcal{F}'$ is a ring of functions on an open ball $\mathbb{B}^n$ of this class.

DEFINITION: **Diffeomorphism** of smooth manifolds is a homeomorphism φ which induces an isomorphims of ringed spaces, that is, φ and φ^{-1} map (locally defined) smooth functions to smooth functions.

Assume from now on that all manifolds are Hausdorff and of class C^∞.

Partition of unity (reminder)

DEFINITION: Let M be a smooth manifold and let $\{U_\alpha\}$ a locally finite cover of M. A **partition of unity** subordinate to the cover $\{U_\alpha\}$ is a family of smooth functions $f_i : M \to [0,1]$ with compact support indexed by the same indices as the U_i's and satisfying the following conditions.
(a) Every function f_i vanishes outside U_i
(b) $\sum_i f_i = 1$

THEOREM: Let $\{U_\alpha\}$ be a countable, locally finite cover of a manifold M, with all U_α diffeomorphic to $\mathbb{R}^n$. **Then there exists a partition of unity subordinate to $\{U_\alpha\}$.**

DEFINITION: Let $U \subset V$ be open subsets in M. We write $U \Subset V$ if the closure of U is contained in V.

DEFINITION: Let $f \in \mathcal{F}(M)$ be a section of a sheaf $\mathcal{F}$ on M. A point $x \in M$ does not lie in the **support** $\operatorname{Sup}(f)$ of f if $f|_U = 0$ for some neighbourhood $U \ni x$.

REMARK: Support of a section is obviously closed.

Smooth functions with prescribed support

EXERCISE: Let $X, Y \subset M$ be non-intersecting closed subsets in a metric space. **Find non-intersecting open neighbourhoods $U_1 \supset X$ and $U_2 \supset U$.**

CLAIM: Let $U \Subset V$ – open subsets in a smooth metrizable manifold. **Then there exists a smooth function $\Phi_{U,V} \in C^\infty M$, supported on V, and equal to 1 on U.**

Proof. Step 1: Find non-intersecting open neighbourhoods U_1 and U_2 of $\overline{U}$ and $M \backslash V$, and choose a partition of unity $\{V_i, \varphi_i\}$ subordinate to the cover $U_1, U_2, U_3 = V \backslash \overline{U}$. **Then for each i, either $\operatorname{Sup}(\varphi_i) \cap U_1 = \emptyset$, or $\operatorname{Sup}(\varphi_i) \cap U_2 = \emptyset$.**

Step 2: Let $\Phi_{U,V} := \sum_S \varphi_i$, where the sum is taken over the set S all φ_i satisfying $\operatorname{Sup}(\varphi_i) \cap U_1 \neq \emptyset$. **Since support of all such φ_i lies in $M \backslash U_2 \subset V$, one has $\operatorname{Sup}(\Phi_{U,V}) \subset V$.** Also, **for each $x \in U_1$, one has $\sum_{i \in S} \varphi_i(x) = 1$, hence $\Phi_{U,V} = 1$ on $U_1 \supset U$.** ■

Vector fields as derivations

DEFINITION: Let M be a smooth manifold. A **vector field** on M is an element in $\operatorname{Der}(C^\infty M)$.

EXAMPLE: For $M = \mathbb{R}^n$, **the space $\operatorname{Der}(C^\infty M)$ is a free module generated by $\frac{d}{dx_i}$, $i = 1, ..., n$.**

REMARK: We want to prove that vector fields form a sheaf. **However, it is not immediately clear how to restrict a vector field from U to $W \subset U$.**

THEOREM: Let $U \Subset V$ be open subset of a smooth metrizable manifold, and $D \in (C^\infty M)$ a derivation. Consider a smooth function $\Phi_{U,V} \in C^\infty M$ supported on V, and equal to 1 on U. Given $f \in C^\infty V$, define $D(f)|_U := D(\Phi_{U,V} f)$. Choosing a cover of such U_i, we can glue together a section $D(f)$ of $C^\infty V$ from such $D(f)\Big|_{U_i}$ **This operation is independent of all choices we made and gives an element $D|_V \in \operatorname{Der}(V)$. Moreover, this restriction maps define a structure of a sheaf on $\operatorname{Der}(M)$.**

Proof: next lecture. The proof uses germs.

Whitney's theorem (with a bound on dimension): strategy of the proof

THEOREM: Let M be a smooth n-manifold. **Then M admits a closed embedding to $\mathbb{R}^{2n+2}$.**

Strategy of the proof:

1. M is embedded to $\mathbb{R}^\infty$.

2. We find a linear projection $\mathbb{R}^\infty \stackrel{\pi}{\longrightarrow} \mathbb{R}^{2n+2}$ such that $\pi|_M$ is a closed embedding of manifolds.

LEMMA: Let $M \subset \mathbb{R}^I$ be a subset, and $\pi: \mathbb{R}^I \longrightarrow \mathbb{R}^J$ a linear projection. Consider the set W of all vectors $\mathbb{R}(x-y)$, where $x, y \in M$ are distinct points. **Then $\pi|_M$ is injective if and only if $\ker \pi \cap W = 0$.**

Proof: $\pi|_M$ is not injective if and only if $\pi(x) = \pi(y)$, which is equivalent to $\pi(x-y) = 0$. ■

Whitney's theorem: injectivity of projections

REMARK: Let $M \subset \mathbb{R}^I$ be a submanifold, and $W \subset \mathbb{R}^I$ the set of all vectors $\mathbb{R}(x-y)$, where $x, y \in M$ are distinct points. **Then W is an image of a $2m+1$-dimensional manifold**, hence (by Sard's Lemma) **for any projection of $\mathbb{R}^I$ to a $(2m+2)$-dimensional space, image of W has measure 0.**

COROLLARY: Let $M \subset \mathbb{R}^I$ be an m-dimensional submanifold, and $S \subset \mathbb{R}^I$ a maximal linear subspace not intersecting W. **Then the projection of W to $\mathbb{R}^I/S$ is surjective.**

Proof: Suppose it's not surjective: $v \notin S$. Then $S \oplus \mathbb{R}v$ satisfies assumptions of lemma, hence $M \longrightarrow \mathbb{R}^I/(S+\mathbb{R}v)$ is also injective. ∎

THEOREM: Let M be a smooth n-manifold, $M \hookrightarrow \mathbb{R}^I$ an embedding constructed earlier. **Then there exists a projection $\pi: \mathbb{R}^I \longrightarrow \mathbb{R}^{2n+2}$ which is injective on M.**

Proof: Let S be the maximal linear subspace such that the restriction of $\pi: \mathbb{R}^I \longrightarrow \mathbb{R}^I/S$ to M is injective. Then the $2m+1$-dimensional manifold W is mapped surjectively to $\mathbb{R}^I/S$, hence $\dim \mathbb{R}^i/S \leqslant 2m+1$ by Sard's lemma. ∎

Tangent space to an embedded manifold

DEFINITION: Let $M \hookrightarrow \mathbb{R}^n$ be a smooth m-submanifold. The **tangent plane** at $p \in M$ is the plane in $\mathbb{R}^n$ tangent to M (i.e, the plane lying in the image of the differential given in local coordinates). A **tangent vector** is an arbitrary vector in this plane with the origin at p. The space of all tangent vectors at p is denoted by T_pM. Given a metric on $\mathbb{R}^n$, we can define the space of **unit tangent vectors** $\mathbb{S}^{m-1}M$ as the set of all pairs (p, v), where $p \in M$, $v \in T_pM$, and $|v| = 1$.

REMARK: $\mathbb{S}^{m-1}M$ is a smooth manifold, projected to M with fibers isomorphic to $m-1$-spheres, hence **$\mathbb{S}^{m-1}M$ is $(2m-1)$-dimensional.**

LEMMA: Let $M \subset \mathbb{R}^I$ be a subset, and $\pi: \mathbb{R}^I \longrightarrow \mathbb{R}^J$ a linear projection. Consider the set W' of all vectors $\mathbb{R}t$, where $t \in T_xM$ **Then the differential $D\pi|_M$ is injective if and only if $\ker\pi \cap W' = 0$.** ■

Now the above argument is repeated: we take a maximal space $S \supset \mathbb{R}^I$ such that the restriction of $\pi: \mathbb{R}^I \longrightarrow \mathbb{R}^I/S$ to M is injective and has injective differential, and the projection of $W \cup W'$ to $\mathbb{R}^I/S$ has to be surjective. However, W' is an image of an $2m$-dimensional manifold $\mathbb{S}^{m-1}M \times \mathbb{R}$, hence **the projection of $W \cup W'$ to $\mathbb{R}^I/S$ can be surjective only if $\dim \mathbb{R}^I/S \leqslant 2m+2$.**

This proves Whitney's theorem.