Geometry of manifolds

Lecture 6: Germs and duality

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Sheaves (reminder)

DEFINITION: An open cover of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$.

REMARK: The definition of a sheaf below is a more abstract version of the notion of "sheaf of functions" defined previously.

DEFINITION: A **presheaf** on a topological space M is a collection of vector spaces $\mathcal{F}(U)$, for each open subset $U \subset M$, together with **restriction maps** $R_{UW}\mathcal{F}(U) \longrightarrow \mathcal{F}(W)$ defined for each $W \subset U$, such that for any three open sets $W \subset V \subset U$, $\Psi_{UW} = \Psi_{UV} \circ \Psi_{VW}$. Elements of $\mathcal{F}(U)$ are called **sections of $\mathcal{F}$ over U**, and restriction map often denoted $f|_W$

DEFINITION: A presheaf $\mathcal{F}$ is called **a sheaf** if for any open set U and any cover $U = \bigcup U_I$ the following two conditions are satisfied.

1. Let $f \in \mathcal{F}(U)$ be a section of $\mathcal{F}$ on U such that its restriction to each U_i vanishes. **Then $f = 0$.**
2. Let $f_i \in \mathcal{F}(U_i)$ be a family of sections compatible on the pairwise intersections: $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

Sheaves and exact sequences (reminder)

DEFINITION: A sequence $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow ...$ of homomorphisms of abelian groups or vector spaces is called **exact** if the image of each map is the kernel of the next one.

CLAIM: A presheaf $\mathcal{F}$ is a sheaf if and only if for every cover $\{U_i\}$ of an open subset $U \subset M$, **the sequence of restriction maps**

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$$

is exact, with $\eta \in \mathcal{F}(U_i)$ mapped to $\eta\Big|_{U_i \cap U_j}$ and $-\eta\Big|_{U_j \cap U_i}$.

Ringed spaces (reminder)

DEFINITION: A sheaf of rings is a sheaf $\mathcal{F}$ such that all the spaces $\mathcal{F}(U)$ are rings, and all restriction maps are ring homomorphisms.

DEFINITION: A sheaf of functions is a subsheaf in a sheaf of all functions, closed under multiplication.

For simplicity, I assume now that a sheaf of rings is a subsheaf in a sheaf of all functions.

DEFINITION: A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of rings. A **morphism** $(M, \mathcal{F}) \xrightarrow{\Psi} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \xrightarrow{\Psi} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $\psi^* f := f \circ \Psi$ belongs to the ring $\mathcal{F}\Big(\Psi^{-1}(U)\Big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

Smooth manifolds (reminder)

DEFINITION: Let $(M, \mathcal{F})$ be a topological manifold equipped with a sheaf of functions. It is said to be a **smooth manifold of class** C^∞ or C^i if every point in $(M, \mathcal{F})$ has an open neighborhood isomorphic to the ringed space $(\mathbb{B}^n, \mathcal{F}')$, where $\mathbb{B}^n \subset \mathbb{R}^n$ is an open ball and $\mathcal{F}'$ is a ring of functions on an open ball $\mathbb{B}^n$ of this class.

DEFINITION: **Diffeomorphism** of smooth manifolds is a homeomorphism φ which induces an isomorphims of ringed spaces, that is, φ and φ^{-1} map (locally defined) smooth functions to smooth functions.

Assume from now on that all manifolds are Hausdorff and of class C^∞.

Partition of unity (reminder)

DEFINITION: Let M be a smooth manifold and let $\{U_\alpha\}$ a locally finite cover of M. A **partition of unity** subordinate to the cover $\{U_\alpha\}$ is a family of smooth functions $f_i : M \to [0,1]$ with compact support indexed by the same indices as the U_i's and satisfying the following conditions.
(a) Every function f_i vanishes outside U_i
(b) $\sum_i f_i = 1$

THEOREM: Let $\{U_\alpha\}$ be a countable, locally finite cover of a manifold M, with all U_α diffeomorphic to $\mathbb{R}^n$. **Then there exists a partition of unity subordinate to $\{U_\alpha\}$.**

DEFINITION: Let $U \subset V$ be open subsets in M. We write $U \Subset V$ if the closure of U is contained in V.

DEFINITION: Let $f \in \mathcal{F}(M)$ be a section of a sheaf $\mathcal{F}$ on M. A point $x \in M$ does not lie in the **support** $\operatorname{Sup}(f)$ of f if $f|_U = 0$ for some neighbourhood $U \ni x$.

REMARK: Support of a section is obviously closed.

Direct limits

DEFINITION: Commutative diagram of vector spaces is given by the following data. There is a directed graph (graph with arrows). For each vertex of this graph we have a vector space, and each arrow corresponds to a homomorphism of the associated vector spaces. **These homomorphism are compatible, in the following way.** Whenever there exist two ways of going from one vertex to another, the compositions of the corresponding arrows are equal.

DEFINITION: Let $\mathcal{C}$ be a commutative diagram of vector spaces, A, B – vector spaces, corresponding to two vertices of a diagram, and $a \in A, b \in B$ elements of these vector spaces. Write $a \sim b$ if a and b are mapped to the same element $d \in D$ by a composition of arrows from $\mathcal{C}$. Let $\sim$ be an equivalence relation generated by such $a \sim b$. A quotient $\bigoplus_i C_i / E$ is called **a direct limit** of a diagram $\{C_i\}$. The same notion is also called **colimit** and **inductive limit**. Direct limit is denoted $\lim\limits_{\to}$.

DEFINITION: Let $\mathcal{F}$ be a sheaf on M, $x \in M$ a point, and $\{U_i\}$ the set of all neighbourhoods of x. Consider a diagram with the set of vertices indexed by $\{U_i\}$, and arrows from U_i to U_j corresponding to inclusions $U_j \hookrightarrow U_i$. The **space of germs** of $\mathcal{F}$ in x is a direct limit $\lim\limits_{\to} \mathcal{F}(U_i)$ over this diagram. The space of germs is also called **a stalk** of a sheaf.

Germs of functions

DEFINITION: A diagram $\mathcal{C}$ is called **filtered** if for any two vertices C_i, C_j, there exists a third vertex C_k, and sequences of arrows leading from C_i to C_k and from C_j to C_k.

EXAMPLE: The diagram formed by all neighbourhoods of a point is obviously filtered.

CLAIM: Let $\mathcal{C}$ be a commutative diagram of vector spaces C_i, with all C_i equipped with a ring structure, and all arrows ring homomorphisms. Suppose that the diagram $\mathcal{C}$ is filtered. **Then there exists a unique ring structure on $C := \varinjlim C_i$ such that all the maps $C_i \longrightarrow C$ are ring homomorphisms.**

DEFINITION: Let $M, \mathcal{F}$ be a ringed space, $x \in M$ its point, and $\{U_i\}$ the set of all its neighbourhoods. Consider a commutative diagram with vertices indexed by $\{U_i\}$, and arrows from U_i to U_j corresponding to inclusions $U_j \hookrightarrow U_i$. For each vertex U_i we take a vector space of sections $\mathcal{F}(U_i)$, and for each arrow the corresponding restriction map. The direct limit of this diagram is called **the ring of germs of the sheaf $\mathcal{F}$ in x**.

Morphisms of sheaves

DEFINITION: Let $\mathcal{B}, \mathcal{B}'$ be sheaves on M. **A sheaf morphism** from $\mathcal{B}$ to $\mathcal{B}'$ is a collection of homomorphisms $\mathcal{B}(U) \longrightarrow \mathcal{B}'(U)$, defined for each open subset $U \subset M$, and compatible with the restriction maps:

$$\begin{array}{ccc} \mathcal{B}(U) & \longrightarrow & \mathcal{B}'(U) \\ \downarrow & & \downarrow \\ \mathcal{B}(U_1) & \longrightarrow & \mathcal{B}'(U_1) \end{array}$$

DEFINITION: A sheaf morphism is called **injective**, or **a monomorphism** if it is injective on stalks and **surjective**, or **epimorphism** if it is surjective on stalks.

EXERCISE: Let $\mathcal{B} \xrightarrow{\varphi} \mathcal{B}'$ be an injective morphism of sheaves on M. **Prove that φ induces an injective map $\mathcal{B}(U) \longrightarrow \mathcal{B}'(U)$ for each U.**

REMARK: A sheaf epimorphism $\mathcal{B} \xrightarrow{\varphi} \mathcal{B}'$ **does not necessarily induce a surjective map $\mathcal{B}(U) \longrightarrow \mathcal{B}'(U)$.**

DEFINITION: **A sheaf isomorphism** is a homomorphism $\Psi: \mathcal{F}_1 \longrightarrow \mathcal{F}_2$, for which there exists an homomorphism $\Phi: \mathcal{F}_2 \longrightarrow \mathcal{F}_1$, such that $\Phi \circ \Psi = \mathrm{Id}$ and $\Psi \circ \Phi = \mathrm{Id}$.

EXERCISE: Show that a morphism of sheaves $\Psi: \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ **is an isomorphism if and only if it is epi and mono.**

Sheaves of modules

REMARK: Let $A: \varphi \longrightarrow B$ be a ring homomorphism, and V a B-module. **Then V is equipped with a natural A-module structure: $av := \varphi(a)v$.**

DEFINITION: Let $\mathcal{F}$ be a sheaf of rings on a topological space M, and $\mathcal{B}$ another sheaf. It is called **a sheaf of $\mathcal{F}$-modules** if for all $U \subset M$ the space of sections $\mathcal{B}(U)$ is equipped with a structure of $\mathcal{F}(U)$-module, and for all $U' \subset U$, the restriction map $\mathcal{B}(U) \xrightarrow{\varphi_{U,U'}} \mathcal{B}(U')$ is a homomorphism of $\mathcal{F}(U)$-modules (use the remark above to obtain a structure of $\mathcal{F}(U)$-module on $\mathcal{B}(U')$).

DEFINITION: A **free sheaf of modules** $\mathcal{F}^n$ over a ring sheaf $\mathcal{F}$ maps an open set U to the space $\mathcal{F}(U)^n$.

DEFINITION: **Locally free sheaf of modules** over a sheaf of rings $\mathcal{F}$ is a sheaf of modules $\mathcal{B}$ satisfying the following condition. For each $x \in M$ there exists a neighbourhood $U \ni x$ such that the restriction $\mathcal{B}|_U$ is free.

DEFINITION: **A vector bundle** on a smooth manifold M is a locally free sheaf of $C^\infty M$-modules.

Dual sheaves

CLAIM: Let $U \subset V$ be open subsets of a Hausdorff space M. **A section $s \in \mathcal{F}(U)$ with compact support $Z \subset U$ can be uniquely extended to $\tilde{s} \in \mathcal{F}(U)$, also with support in Z.**

Proof: Z is a closed subset of U, not intersecting $M \backslash V$. Let U_1 be an open neighbourhood of Z not intersecting $M \backslash V$, and $U_2 := M \backslash Z$. Then $\{U_1, U_2\}$ is a cover of M, and $s\big|_{U_1 \cap U_2} = 0$, hence **$\tilde{s}$ can be glued from $s \in \mathcal{F}(U_1)$ and $0 \in \mathcal{F}(U_2)$.** ■

DEFINITION: Let $\mathcal{F}$ be a sheaf. Denote the space of sections of $\mathcal{F}$ on U with compact support by $\mathcal{F}_c(U)$. Let $\mathcal{F}^*(U)$ map U to the dual space $\mathcal{F}_c(U)^*$. Using the claim above, we obtain a restriction map $\mathcal{F}^*(V) \longrightarrow \mathcal{F}^*(U)$ for each open $V \supset U$. This gives **dual presheaf** $\mathcal{F}^*$

EXERCISE: Let M be a manifold, and $\mathcal{F}$ a sheaf of modules over $C^\infty M$. **Prove that $\mathcal{F}^*$ is a sheaf.**

HINT: Use partition of unity.

Rings and derivations (reminder)

REMARK: All rings in these handouts are assumed to be commutative and with unit. Algebras are associative, but not necessarily commutative (such as the matrix algebra). **Rings over a field k** are rings containing a field k. We assume that k has characteristic 0.

DEFINITION: Let R be a ring over a field k. A k-linear map $D\ R \longrightarrow R$ is called **a derivation** if it satisfies **the Leibnitz equation** $D(fg) = D(f)g + gD(f)$. The space of derivations is denoted as $\operatorname{Der}_k(R)$.

EXAMPLE: $\frac{d}{dt}:\ \mathbb{C}[t] \longrightarrow \mathbb{C}[t]$. $\frac{d}{dt}:\ C^\infty\mathbb{R} \longrightarrow C^\infty\mathbb{R}$.

THEOREM: Let $x_1, ..., x_n$ be coordinate functions on $\mathbb{R}^n$, $R = C^\infty\mathbb{R}^n$, and $\operatorname{Der}(R) \xrightarrow{\Pi} (C^\infty\mathbb{R}^n)^n$ map D to $(D(x_1), D(x_2), ..., D(x_n))$. **Then**

$$\Pi :\ \operatorname{Der}(C^\infty\mathbb{R}^n) \longrightarrow R^n$$

is an isomorphism.

Derivations of an algebra of compactly supported functions

SUBLEMMA: Let M be a smooth manifold, and $C^\infty_c(U)$ the space of compactly supported smooth functions, equipped with natural multiplication. Consider the space Der$(M, C^\infty_c(M))$ of derivations of $C^\infty_c(M)$ with values in $C^\infty(M)$. Let f be a function with support in $Z \subset M$. **Then $D(f)$ has support in Z for each $D \in Der(M, C^\infty_c(M))$.**

Proof: For each g with support outside of Z, we have $0 = D(fg) = fD(g) + gD(f)$, hence for each $U \subset M\backslash Z$, we have $0 = gD(f)|_U$ whenever Sup$(g) \subset U$. Then, $D(f)|_U = 0$ as well. ■

LEMMA: Let φ be a smooth function with compact support which is equal to 1 on $U \subset M$, and $D \in C^\infty_c(M)$. Then $D(\varphi f)|_U = D(f)|_U$.

Proof. Step 1: $D(\varphi f) = D(\varphi)f - \varphi D(f)$, hence **it would suffice to prove that $D(\varphi)|_U = 0$.**

Step 2: Now, $\varphi^2 - \varphi = 0$ on U. By the Sublemma above, $0 = D(\varphi^2 - \varphi)|_U = [2D(\varphi)\varphi - D(\varphi)]|_U = D(\varphi)|_U$. ■

Derivations with compact support and without

THEOREM: Define the tautological map $\operatorname{Der}(C^\infty(M)) \xrightarrow{\tau} \operatorname{Der}(M, C^\infty_c(M))$, taking D to itself. **Then τ is an isomorphism:**

$$\operatorname{Der}(M, C^\infty_c(M)) = \operatorname{Der}(C^\infty(M)).$$

Proof. Step 1: For each $x \in M$, consider a smooth function ψ_x with compact support which is equal to 1 in some neighbourhood $U_x \ni x$. Let $D \in \ker\tau$, and $f \in C^\infty M$. Then $0 = D(\psi_x f)\Big|_{U_x} = D(f)\Big|_{U_x}$, hence $D(f) = 0$ in a neighbourhood of any $x \in M$. **This implies that $\ker\tau = 0$.**

Step 2: Consider $D \in \operatorname{Der}(M, C^\infty_c(M))$, and let $D(f)\Big|_{U_x} := D(\psi_x f)$. For any U_x, U_y, the Lemma above implies

$$D(\psi_x f)\Big|_{U_x \cap U_y} = D(\psi_y f)\Big|_{U_x \cap U_y} = D(f)\Big|_{U_x \cap U_y}$$

Therefore, the sections $D(f)\Big|_{U_x}$ agree on pairwise intersections, and define a section $\tilde{D}(f) \in C^\infty M$.

Step 3: On germs, $\tilde{D}(f) = D(f)$, hence it satisfies Leibnitz rule on each germ, and therefore, **$\tilde{D}$ is a derivation (exercise: check this directly).**

Step 4: By the Lemma above, $\tilde{D}(f)\Big|_{U_x} = D(f)\Big|_{U_x}$ for each f with compact support. **Therefore, $\tau(\tilde{D}) = D$, and τ is surjective.** ■

Derivations as a sheaf

DEFINITION: Let $U \subset V$ be open subsets of a smooth manifold M, and $D \in \operatorname{Der}(V, C_c^\infty(V))$. For any $f \in C_c^\infty(U)$, extend f to $\tilde{f} \in C_c^\infty(V)$ with the same support, (zero extension), and let $D|_U(f) := D(\tilde{f})$. **This defines a structure of presheaf $U \longrightarrow C_c^\infty(U)$.**

CLAIM: $U \longrightarrow C_c^\infty(U)$ is a sheaf.

Proof. Step 1: A vector field is uniquely determined by its restriction to the germs of all sections, hence a derivation D which vanishes on all germs for all $x \in M$ vanishes everywhere. **This takes care of the first sheaf axiom.**

Proof. Step 2: Let $\{U_i\}$ be a cover of M. To glue a derivation D from its bits $D_i \in \operatorname{Der}(C^\infty(U_i))$, consider a partition of unity ψ_i subordinate to $\{U_i\}$. **Then $D(f) := \sum D_i(\psi_i f)$ is a derivation which restricts to all D_i.** ■

COROLLARY: The sheaf of derivations is locally free, that is, $\operatorname{Der} C^\infty M$ defines a vector bundle on M.

DEFINITION: It is called **the tangent bundle**, and denoted TM.

NEXT TIME.

Next time:

TEST ASSIGNMENT # 2!