Geometry of manifolds

Lecture 7: Categories and locally trivial fibrations

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Locally trivial fibrations

DEFINITION: A smooth map $f: X \longrightarrow Y$ is called **a locally trivial fibration** if each point $y \in Y$ has a neighbourhood $U \ni y$ such that $f^{-1}(U)$ is diffeomorphic to $U \times F$, and the map $f: f^{-1}(U) = U \times F \longrightarrow U$ is a projection. In such situation, F is called **the fiber** of a locally trivial fibration.

DEFINITION: **A trivial fibration** is a map $X \times Y \longrightarrow Y$.

EXAMPLE: The projection $S^3 \subset \mathbb{C}^2 \backslash 0 \xrightarrow{f} \mathbb{C}P^1$ is called **the Hopf fibration**. Given $U = \{x : 1\} \subset \mathbb{C}P^1$, with $|x| \leqslant 1$, one has

$$f^{-1}(U) = \{z_1, z_2 \in S^3 \ \ | \ \ |z_1|^2 + |z_2|^2 = 1, |z_1| \leqslant 1\}$$

(here z_i are complex coordinates in $\mathbb{C}^2$). Then

$$f^{-1}(U) = \left\{(z_1, z_2) \ \ | \ \ z_2 \in U(1) \cdot \sqrt{1 - |z_1|^2}\right\},$$

where $U(1) = \{z \in \mathbb{C} \ \ | \ \ |z| = 1\}$. Therefore, **the Hopf fibration $f: S^3 \longrightarrow S^2$ is a locally trivial fibration**.

REMARK: Since $\pi_1(S^3) = 0$ and $\pi_1(S^1 \times S^2) = \mathbb{Z}$, **the Hopf fibration is non-trivial.**

Vector bundles

DEFINITION: A vector bundle on Y is a locally trivial fibration $f: X \longrightarrow Y$ with fiber $\mathbb{R}^n$, with each fiber $V := f^{-1}(y)$ equipped with a structure of a vector space, smoothly depending on $y \in Y$.

REMARK: This definition **is not very precise or rigorous,** because "smoothly depending on $y \in Y$" **needs to be explained.**

REMARK: This definition is compatible with the one we used previously ("a vector bundle is a locally free sheaf of $C^\infty M$-modules"). This will be explained later.

For a more rigorous approach:

1. Define categories.

2. Define group objects and vector space objects

3. Formulate "smoothly depending on $y \in Y$" in these terms.

Categories: data

DEFINITION: A category $\mathcal{C}$ is a collection of data called "objects" and "morphisms between objects" which satisfies the axioms below.

DATA.

Objects: The set $\mathcal{O}b(\mathcal{C})$ of **objects** of $\mathcal{C}$.

Morphisms: For each $X, Y \in \mathcal{O}b(\mathcal{C})$, one has a set $\mathcal{M}or(X, Y)$ of **morphisms from X to Y**.

Composition of morphisms: For each $\varphi \in \mathcal{M}or(X, Y), \psi \in \mathcal{M}or(Y, Z)$ there exists **the composition** $\varphi \circ \psi \in \mathcal{M}or(X, Z)$

Identity morphism: For each $A \in \mathcal{O}b(\mathcal{C})$ there exists a morphism $\mathrm{Id}_A \in \mathcal{M}or(A, A)$.

REMARK: In some versions of axiomatic set theory, one considers not a set, but **a class** of objects, which could be arbitrarily big, such as the class of all sets, or the class of all linear spaces. The category with **a set** of morphisms and objects is called **a small category**, and one with a class **a big category**.

In ZFC, one postulates existence of so-called **Grothendieck universe** (that is, **a strongly inaccessible cardinal**). **Small sets** are ones which belong to the Grothendieck universe, the rest of the sets are **big**.

Existence of a strongly inaccessible cardinal **implies consistency of ZFC** (Goedel).

Categories: axioms

AXIOMS.

Associativity of composition: $\varphi_1 \circ (\varphi_2 \circ \varphi_3) = (\varphi_1 \circ \varphi_2) \circ \varphi_3$.

Properties of identity morphism: For each $\varphi \in \mathcal{M}\mathrm{or}(X, Y)$, one has $\mathrm{Id}_x \circ \varphi = \varphi = \varphi \circ \mathrm{Id}_Y$

DEFINITION: Let $X, Y \in \mathcal{O}\mathrm{b}(\mathcal{C})$ – objects of $\mathcal{C}$. A morphism $\varphi \in \mathcal{M}\mathrm{or}(X, Y)$ is called **an isomorphism** if there exists $\psi \in \mathcal{M}\mathrm{or}(Y, X)$ such that $\varphi \circ \psi = \mathrm{Id}_X$ and $\psi \circ \varphi = \mathrm{Id}_Y$. In this case, the objects X and Y are called **isomorphic**.

Examples of categories:

Category of sets: its morphisms are arbitrary maps.

Category of vector spaces: its morphisms are linear maps.

Categories of rings, groups, fields: morphisms are homomorphisms.

Category of topological spaces: morphisms are continuous maps.

Category of smooth manifolds: morphisms are smooth maps.

It is often convenient to express morphisms by arrows, and call them "maps".

Some categorical constructions

DEFINITION: **A product** $X_1 \times X_2$ of $X_1, X_2 \in \mathcal{O}\mathrm{b}(cac)$ is an object of $\mathcal{C}$ equipped with **projection maps** $\pi_i : X_1 \times X_2 \longrightarrow X_i$ such that **for any pair of morphisms** $\varphi_i \in \mathcal{M}\mathrm{or}(Y, X_i)$ **there is a unique morphism** $\varphi \in \mathcal{M}\mathrm{or}(Y, X_1 \times X_2)$ **such that** $\varphi \circ \pi_i = \varphi_i$**.**

EXERCISE: Prove that **a product is unique up to isomorphism,** if it exists.

EXERCISE: Prove that the product is **associative:** $X \times (Y \times Z) \cong (X \times Y) \times Z$ and **commutative:** $X \times Y \cong Y \times X$.

EXERCISE: Find the product in the categories of a. groups b. rings c. vector spaces d. sets e. topological spaces.

DEFINITION: **An initial object** of a category is an object $I \in \mathcal{O}\mathrm{b}(\mathcal{C})$ such that $\mathcal{M}\mathrm{or}(I, X)$ is always a set of one element. **A terminal object** is $T \in \mathcal{O}\mathrm{b}(\mathcal{C})$ such that $\mathcal{M}\mathrm{or}(X, T)$ is always a set of one element.

EXERCISE: Prove that **the initial and the terminal object is unique,** up to isomorphism.

Group objects in categories

EXERCISE: Let T be a terminal object. **Prove that $X \times T \cong X$ for each $X \in \mathcal{O}b(\mathcal{C})$.**

DEFINITION: An object $G \in \mathcal{O}b(\mathcal{C})$ is called **a group object** if there exists a morphism $\mu \in \mathcal{M}or(G \times G, G)$ (**the product**), a morphism $e \in \mathcal{M}or(T, G)$ from the terminal object (**the unit**), and a morphism $i \in \mathcal{M}or(G, G)$ (**the inverse**), satisfying the following axioms.

Associativity: Consider the morphisms $\mu_{12}, \mu_{23}: G \times G \times G \longrightarrow G \times G$, the first map takes the product on the first two objects, and acts as identity on the third, the second maps is a product on last 2 and identity on the first. Then $\mu_{12} \circ \mu = \mu_{23} \circ \mu : G \times G \times G \longrightarrow G$.

Unit: The compositions $G = G \times T \xrightarrow{\mathrm{Id}_G \times e} G \times G \xrightarrow{\mu} G$ and $G = G \times T \xrightarrow{e \times \mathrm{Id}_G} G \times G \xrightarrow{\mu} G$ are identities.

Inverse: Let $\Delta : G \longrightarrow G \times G$ be **the diagonal map,** that is, a map $G \longrightarrow G \times G$ obtained from a pair of identity maps. Then the composition $G \xrightarrow{\Delta} G \times G \xrightarrow{\mathrm{Id}_G \times i} G \times G \xrightarrow{\mu} G$ is equal to $G \longrightarrow T \xrightarrow{e} G$.

Examples of group objects

EXAMPLE: A topological group is a group object in the category of topological spaces.

EXAMPLE: A Lie group is a group object in the category of smooth manifolds.

DEFINITION: Let $\mathcal{C}$ be a category. **An opposite category** $\mathcal{C}^\circ$ is a category with the same sets of objects, $\mathcal{M}\mathrm{or}_{\mathcal{C}^o}(X,Y) = \mathcal{M}\mathrm{or}_{\mathcal{C}}(Y,X)$, with the same compositions as in $\mathcal{C}$ taken in inverse order.

EXAMPLE: The category of finitely generated algebras withous nilpotents over $\mathbb{C}$ is equivalent to $\mathcal{C}^\circ$, where $\mathcal{C}$ is a category with objects algebraic subsets in $\mathbb{C}^n$ (common zeros of a system of polynomial equations) and morphisms polynomial functions. This statement is called **"Hilbert's Nullstellensatz".**

EXAMPLE: An algebraic group is a group object in the category $\mathcal{C}^\circ$, where $\mathcal{C}$ is a category of rings.

EXAMPLE: A formal group is a group object in the category $\mathcal{C}^o$, where $\mathcal{C}$ is a category of complete local rings (over $\mathbb{C}$, these are local rings, obtained as quotients of the ring of formal power series by an ideal).

Topological groups over a base

DEFINITION: Fix a topological space M, and let $\mathcal{C}_M$ be a category of pairs $(X, f : X \longrightarrow M)$ with morphisms being continuous maps from X_1 to X_2 commuting with the projections to M. The product in $\mathcal{C}_M$ is called **fiber product:** $X_1 \times_M X_2 := \{(x_1, x_2) \in X_1 \times X_2 \mid f_1(x_1) = f_2(x_2)\}$. A group object in $\mathcal{C}_M$ is called **a topological group over M**.

REMARK: This definition is equivalent to the following.

DEFINITION: Let $B \xrightarrow{\pi} M$ be a continuous map, and $B \times_M B \xrightarrow{\Psi} M$ - a morphism over M. This morphism is called **associative multiplication** if it is associative on the fibers of π, that is, satisfies $\Psi(a, \Psi(b,c)) = \Psi(\Psi(a,b),c)$ for every triple a, b, c in the same fiber.

A section $M \xrightarrow{e} B$ is called **the unit** if the maps $B \xrightarrow{\mathrm{Id}_B \times e} B \times_M B \xrightarrow{\Psi} B$ and $B \xrightarrow{e \times \mathrm{Id}_B} B \times_M B \xrightarrow{\Psi} B$ are equal to Id_B.

A morphism $\nu : B \longrightarrow B$ over M is called **a group inverse** if each of the maps $B \xrightarrow{\Delta} B \times_M B \xrightarrow{\mathrm{Id}_B \times \nu} B \times_M B \xrightarrow{\Psi} B$ and $B \xrightarrow{\Delta} B \times_M B \xrightarrow{\nu \times \mathrm{Id}_B} B \times_M B \xrightarrow{\Psi} B$ is a constant map, mapping b to $e(\pi(b))$.

A map $B \xrightarrow{\pi} M$ equipped with associative multiplication, unit and group inverse is called **a topological group over M**.

Vector spaces over a base

REMARK: Let $\pi : G \longrightarrow M$ be a topological group over M. Then the fiber $\pi^{-1}(m)$ is a group for each $m \in M$. **This group structure depends on $m \in M$ continuously**, but **to state this dependency formaly, one needs to define a topological group over M.**

DEFINITION: Let k be a field. **A k-vector space object** in a category $\mathcal{C}$ is a group object V equipped with a set of morphisms $\lambda_x \in \mathcal{M}\mathrm{or}(V,V)$, parametrized by $x \in k$, and satisfying the following conditions.

Multiplicativity: $\lambda_x \lambda_y = \lambda_{xy}$,

Zero: $\lambda_1 = \mathrm{Id}_V$

Unit: $\lambda_0 : V \longrightarrow V$ is a composition $V \longrightarrow T \xrightarrow{e} V$.

Additivity: Let Δ be the diagonal map. Then the composition $G \xrightarrow{\Delta} G \times G \longrightarrow \lambda_x \times \lambda_y \xrightarrow{\mu} G$ is equal to λ_{x+y}.

Distributivity: The composition $G \times G \xrightarrow{\lambda_x \times \lambda_x} G \times G \xrightarrow{\mu} G$ is equal to $\mu \circ \lambda_x$.

DEFINITION: Let k be a topological field (for instance, $\mathbb{C}$ or $\mathbb{R}$). **A topological vector space B over a base M** is a vector space object in $\mathcal{C}_M$, such that the map $\lambda_x : k \times B \longrightarrow B$ is continuous.

Vector spaces over a base (category-free definition)

DEFINITION: Let G be an abelian group, and k a field. Suppose that for each non-zero $\lambda \in k$ there exists an automorphism $\varphi_\lambda : G \longrightarrow G$, such that $\varphi_\lambda \circ \varphi_{\lambda'} = \varphi_{\lambda\lambda'}$, and $\varphi_{\lambda+\lambda'}(g) = \varphi_\lambda(g) + \varphi_{\lambda'}(g)$. Then G is called **a vector space over k**.

DEFINITION: Let $k = \mathbb{R}$ or $\mathbb{C}$. An abelian topological group $B \xrightarrow{\pi} M$ over M is called **a vector space over a base M**, or **a relative vector space over M** if for each non-zero $\lambda \in k$ there exists a continuous automorphism $\varphi_\lambda : B \longrightarrow B$ of a group B over M satisfying assumptions of the above definition.

REMARK: Let $B \xrightarrow{\pi} M$ be a relative vector space over M, $U \subset M$ an open subset, and $\mathcal{B}(U)$ the space of sections of a map $\pi^{-1}(U) \xrightarrow{\pi} U$. Then **$\mathcal{B}(U)$ defines a sheaf of modules over a sheaf $C^0(M)$ of continuous functions.**

EXAMPLE: Let $S \subset \mathbb{R}^n$ be a subset (not necessarily a smooth submanifold), $s \in S$ a point, and $v \in T_s\mathbb{R}^n$ a vector. We sat that v belongs to a **tangent cone** C_sS if the distance from S to a point $s+tv$ converges to 0 as $t \to 0$ faster than linearly: $\lim_{t\to 0} \frac{d(S,s+tv)}{t} = 0$. **Then the set CS of all pairs $(s,v), s \in S, v \in C_sS$ is a relative vector space over S.**

Total space of a vector bundle

DEFINITION: Let $B \longrightarrow M$ be a smooth locally trivial fibration with fiber $\mathbb{R}^n$. Assume that B is equipped with a structure of relative vector space over M, and all the maps used in the definition of a relative vector space are smooth. Then B is called **a total space of a vector bundle.**

REMARK: Let $\pi: B \longrightarrow M$ be a total space of a vector bundle, $U \subset M$ open subset, and $\mathcal{B}(U)$ the space of all smooth sections of $\pi^{-1}(U) \xrightarrow{\pi} U$. **Then $\mathcal{B}$ is a locally free sheaf of $C^\infty M$-modules**.

THEOREM: Every locally free sheaf $C^\infty M$-modules is defined from a total space of a vector bundle, which is determined uniquely by a sheaf.

The proof will be a couple of slides below.

Fiber of a locally free sheaf

DEFINITION: Let $\mathcal{B}$ be an n-dimensional locally free sheaf of C^∞-modules on M, $x \in M$ a point, $\mathcal{B}_x$ the space of germs of $\mathcal{B}$ in x, and $\mathfrak{m}_x \subset C^\infty_x M$ the maximal ideal in the ring of germs $C^\infty_x M$ of smooth functions. Define **the fiber** of $\mathcal{B}$ in x as a quotient $\mathcal{B}_x / \mathfrak{m}_x \mathcal{B}_x$. A fiber of $\mathcal{B}$ is denoted $\mathcal{B}|_x$.

REMARK: A fiber of an n-dimensional bundle is an n-dimensional vector space.

REMARK: Let $\mathcal{B} = C^\infty M^n$, and $b \in \mathcal{B}|_x$ a point of a fiber, represented by a germ $\varphi \in \mathcal{B}_x = C^\infty_m M^n$, $\varphi = (f_1, ..., f_n)$. Consider a map Ψ from the set of all fibers $\mathcal{B}$ to $M \times \mathbb{R}^n$, mapping $(x, \varphi = (f_1, ..., f_n))$ to $(f_1(x), ..., f_n(x))$. **Then Ψ is bijective.** Indeed, $\mathcal{B}|_x = \mathbb{R}^n$.

Total space of a vector bundle from its sheaf of sections

DEFINITION: Let $\mathcal{B}$ be an n-dimensional locally free sheaf of C^∞-modules. Denote the set of all vectors in all fibers of $\mathcal{B}$ over all points of M by $\operatorname{Tot}\mathcal{B}$. Let $U \subset M$ be an open subset of M, with $\mathcal{B}|_U$ a trivial bundle. Using the local bijection $\operatorname{Tot}\mathcal{B}(U) = U \times \mathbb{R}^n$ we consider topology on $\operatorname{Tot}\mathcal{B}$ induced by open subsets in $\operatorname{Tot}\mathcal{B}(U) = U \times \mathbb{R}^n$ for all open subsets $U \subset M$ and all trivializations of $\mathcal{B}|_U$. Then $\operatorname{Tot}\mathcal{B}$ is called **a total space of a vector bundle $\mathcal{B}$.**

CLAIM: The space $\operatorname{Tot}\mathcal{B}$ with this topology **is a locally trivial fibration over M, with fiber $\mathbb{R}^n$.** Moreover, it is a relative vector space over M, and **the sheaf of smooth sections of $\operatorname{Tot}\mathcal{B} \longrightarrow M$ is isomorphic to $\mathcal{B}$.**

REMARK: This gives an equivalence between locally free sheaves of $\mathbb{C}^\infty$-modules and the total spaces of vector bundles, defined abstractly in terms of locally trivial fibrations.