Geometry of manifolds

Lecture 8: Vector bundles and locally trivial fibrations

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Locally trivial fibrations

DEFINITION: A smooth map $f: X \longrightarrow Y$ is called **a locally trivial fibration** if each point $y \in Y$ has a neighbourhood $U \ni y$ such that $f^{-1}(U)$ is diffeomorphic to $U \times F$, and the map $f: f^{-1}(U) = U \times F \longrightarrow U$ is a projection. In such situation, F is called **the fiber** of a locally trivial fibration.

DEFINITION: A trivial fibration is a map $X \times Y \longrightarrow Y$.

DEFINITION: A vector bundle on Y is a locally trivial fibration $f: X \longrightarrow Y$ with fiber $\mathbb{R}^n$, with each fiber $V := f^{-1}(y)$ equipped with a structure of a vector space, smoothly depending on $y \in Y$.

REMARK: This definition **is not very precise or rigorous,** because "smoothly depending on $y \in Y$" **needs to be explained.**

REMARK: This definition is compatible with the one we used previously ("a vector bundle is a locally free sheaf of $C^\infty M$-modules"). This will be explained later today.

Fiber product

DEFINITION: Fix a topological space M, and let $\pi_1 : X_1 \longrightarrow M$ and $\pi_2 : X_2 \longrightarrow M$ be continuous maps. The **fiber product** is defined as

$$X_1 \times_M X_2 := \{(x_1, x_2) \in X_1 \times X_2 \mid \pi_1(x_1) = \pi_2(x_2)\}.$$

REMARK: Consider the projection $X_1 \times_M X_2 \xrightarrow{\pi} M$. Then

$$\pi^{-1}(m) = \pi_1^{-1}(m) \times \pi_2^{-1}(m).$$

EXERCISE: Let $\pi_i : X_i \longrightarrow M$, $i = 1, 2$ be trivial fibrations, $X_i = M \times F_i$. **Prove that $X_1 \times_M X_2 = M \times F_1 \times F_2$.**

REMARK: If X_i are locally trivial fibrations over M with fiber F_i, **the fiber product $X_1 \times_M X_2$ is a locally trivial fibration over M with fiber $F_1 \times F_2$.**

Fiber product: universal property

REMARK: The fiber product satisfies the following **universal property**. Let $X_1 \times_M X_2 \xrightarrow{\tilde{\pi}_i} X_i$ be the natural projection. Then any commutative square

$$\begin{array}{ccc} Y & \xrightarrow{f_1} & X_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ X_2 & \xrightarrow{\pi_2} & M \end{array}$$

induces a unique continuous map $f: Y \longrightarrow X_1 \times_M X_2$ such that $f \circ \tilde{\pi}_i = f_i$. Moreover, **any space satisfying the universal property is homeomorphic to $X_1 \times_M X_2$.**

REMARK: This statement is awkward, because I avoided the language of categories.

EXERCISE: Translate this property to the language of categories.

Topological groups over a base

DEFINITION: Fix a topological space M. **A space over M** $(X, f: X \longrightarrow M)$ is a topological space equipped with a continuous map to M. **A morphisms** $\varphi: (X_1, f_1) \longrightarrow (X_2, f_2)$ is a continuous map $\varphi: X_1 \longrightarrow X_2$ commuting with projections to M.

DEFINITION: Let $B \stackrel{\pi}{\longrightarrow} M$ be a continuous map, and $B \times_M B \stackrel{\Psi}{\longrightarrow} M$ - a morphism over M. This morphism is called **associative multiplication** if it is associative on the fibers of π, that is, satisfies $\Psi(a, \Psi(b, c)) = \Psi(\Psi(a, b), c)$ for every triple a, b, c in the same fiber.

A section $M \stackrel{e}{\longrightarrow} B$ is called **the unit** if the maps $B \stackrel{\mathrm{Id}_B \times e}{\longrightarrow} B \times_M B \stackrel{\Psi}{\longrightarrow} B$ and $B \stackrel{e \times \mathrm{Id}_B}{\longrightarrow} B \times_M B \stackrel{\Psi}{\longrightarrow} B$ are equal to Id_B.

A morphism $\nu: B \longrightarrow B$ over M is called **a group inverse** if each of the maps $B \stackrel{\Delta}{\longrightarrow} B \times_M B \stackrel{\mathrm{Id}_B \times \nu}{\longrightarrow} B \times_M B \stackrel{\Psi}{\longrightarrow} B$ and $B \stackrel{\Delta}{\longrightarrow} B \times_M B \stackrel{\nu \times \mathrm{Id}_B}{\longrightarrow} B \times_M B \stackrel{\Psi}{\longrightarrow} B$ is a constant map, mapping b to $e(\pi(b))$.

A map $B \stackrel{\pi}{\longrightarrow} M$ equipped with associative multiplication, unit and group inverse is called **a topological group over M**.

Vector spaces over a base

DEFINITION: Let G be an abelian group, and k a field. Suppose that for each non-zero $\lambda \in k$ there exists an automorphism $\varphi_\lambda : G \longrightarrow G$, such that $\varphi_\lambda \circ \varphi_{\lambda'} = \varphi_{\lambda\lambda'}$, and $\varphi_{\lambda+\lambda'}(g) = \varphi_\lambda(g) + \varphi_{\lambda'}(g)$. Then G is called **a vector space over k**.

DEFINITION: Let $k = \mathbb{R}$ or $\mathbb{C}$. An abelian topological group $B \xrightarrow{\pi} M$ over M is called **a vector space over a base M**, or **a relative vector space over M** if for each non-zero $\lambda \in k$ there exists a continuous automorphism $\varphi_\lambda : B \longrightarrow B$ of a group B over M satisfying assumptions of the above definition.

REMARK: Let $B \xrightarrow{\pi} M$ be a relative vector space over M, $U \subset M$ an open subset, and $\mathcal{B}(U)$ the space of sections of a map $\pi^{-1}(U) \xrightarrow{\pi} U$. Then **$\mathcal{B}(U)$ defines a sheaf of modules over a sheaf $C^0(M)$ of continuous functions.**

EXAMPLE: Let $S \subset \mathbb{R}^n$ be a subset (not necessarily a smooth submanifold), $s \in S$ a point, and $v \in T_s\mathbb{R}^n$ a vector. We sat that v belongs to a **tangent cone** C_sS if the distance from S to a point $s+tv$ converges to 0 as $t \to 0$ faster than linearly: $\lim_{t\to 0} \frac{d(S,s+tv)}{t} = 0$. **Then the set CS of all pairs $(s,v), s \in S, v \in C_sS$ is a relative vector space over S.**

Total space of a vector bundle

DEFINITION: Let $B \longrightarrow M$ be a smooth locally trivial fibration with fiber $\mathbb{R}^n$. Assume that B is equipped with a structure of relative vector space over M, and all the maps used in the definition of a relative vector space are smooth. Then B is called **a total space of a vector bundle.**

REMARK: Let $\pi: B \longrightarrow M$ be a total space of a vector bundle, $U \subset M$ open subset, and $\mathcal{B}(U)$ the space of all smooth sections of $\pi^{-1}(U) \xrightarrow{\pi} U$. **Then $\mathcal{B}$ is a locally free sheaf of $C^\infty M$-modules**.

THEOREM: Every locally free sheaf $C^\infty M$-modules is defined from a total space of a vector bundle, which is determined uniquely by a sheaf.

The proof will be a couple of slides below.

Fiber of a locally free sheaf

DEFINITION: Let $\mathcal{B}$ be an n-dimensional locally free sheaf of C^∞-modules on M, $x \in M$ a point, $\mathcal{B}_x$ the space of germs of $\mathcal{B}$ in x, and $\mathfrak{m}_x \subset C^\infty_x M$ the maximal ideal in the ring of germs $C^\infty_x M$ of smooth functions. Define **the fiber** of $\mathcal{B}$ in x as a quotient $\mathcal{B}_x/\mathfrak{m}_x\mathcal{B}_x$. A fiber of $\mathcal{B}$ is denoted $\mathcal{B}|_x$.

REMARK: A fiber of an n-dimensional bundle is an n-dimensional vector space.

REMARK: Let $\mathcal{B} = C^\infty M^n$, and $b \in \mathcal{B}|_x$ a point of a fiber, represented by a germ $\varphi \in \mathcal{B}_x = C^\infty_m M^n$, $\varphi = (f_1, ..., f_n)$. Consider a map Ψ from the set of all fibers $\mathcal{B}$ to $M \times \mathbb{R}^n$, mapping $(x, \varphi = (f_1, ..., f_n))$ to $(f_1(x), ..., f_n(x))$. **Then Ψ is bijective.** Indeed, $\mathcal{B}|_x = \mathbb{R}^n$.

Total space of a vector bundle from its sheaf of sections

DEFINITION: Let $\mathcal{B}$ be an n-dimensional locally free sheaf of C^∞-modules. Denote the set of all vectors in all fibers of $\mathcal{B}$ over all points of M by $\operatorname{Tot}\mathcal{B}$. Let $U \subset M$ be an open subset of M, with $\mathcal{B}|_U$ a trivial bundle. Using the local bijection $\operatorname{Tot}\mathcal{B}(U) = U \times \mathbb{R}^n$ we consider topology on $\operatorname{Tot}\mathcal{B}$ induced by open subsets in $\operatorname{Tot}\mathcal{B}(U) = U \times \mathbb{R}^n$ for all open subsets $U \subset M$ and all trivializations of $\mathcal{B}|_U$. Then $\operatorname{Tot}\mathcal{B}$ is called **a total space of a vector bundle $\mathcal{B}$.**

CLAIM: The space $\operatorname{Tot}\mathcal{B}$ with this topology **is a locally trivial fibration over M, with fiber $\mathbb{R}^n$.** Moreover, it is a relative vector space over M, and **the sheaf of smooth sections of $\operatorname{Tot}\mathcal{B} \longrightarrow M$ is isomorphic to $\mathcal{B}$.**

REMARK: This gives an equivalence between locally free sheaves of $\mathbb{C}^\infty$-modules and the total spaces of vector bundles, defined abstractly in terms of locally trivial fibrations.

Tensor product

DEFINITION: Let V, V' be R-modules, W a free abelian group generated by $v \otimes v'$, with $v \in V, v' \in V'$, and $W_1 \subset W$ a subgroup generated by combinations $rv \otimes v' - v \otimes rv'$, $(v_1 + v_2) \otimes v' - v_1 \otimes v' - v_2 \otimes v'$ and $v \otimes (v'_1 + v'_2) - v \otimes v'_1 - v \otimes v'_2$. Define **the tensor product** $V \otimes_R V'$ as a quotient group W/W_1.

EXERCISE: Show that $r \cdot v \otimes v' \mapsto (rv) \otimes v'$ **defines an R-module structure on $V \otimes_R V'$.**

REMARK: Let $\mathcal{F}$ be a sheaf of rings, and $\mathcal{B}_1$ and $\mathcal{B}_2$ be sheaves of locally free $(M, \mathcal{F})$-modules. **Then**

$$U \longrightarrow \mathcal{B}_1(U) \otimes_{\mathcal{F}(U)} \mathcal{B}_2(U)$$

is also a locally free sheaf of modules.

DEFINITION: **Tensor product** of vector bundles is a tensor product of the corresponding sheaves of modules.

Dual bundle and bilinear forms

DEFINITION: Let V be an R-module. **A dual R-module** V^* is $\operatorname{Hom}_R(V,R)$ with the R-module structure defined as follows: $r \cdot h(\dots) \mapsto rh(\dots)$.

CLAIM: Let $\mathcal{B}$ be a vector bundle, that is, a locally free sheaf of $C^\infty M$-modules, and $\operatorname{Tot}\mathcal{B} \xrightarrow{\pi} M$ its total space. Define $\mathcal{B}^*(U)$ as a space of smooth functions on $\pi^{-1}(U)$ linear in the fibers of π. **Then $\mathcal{B}^*(U)$ is a locally free sheaf over $C^\infty(M)$.**

DEFINITION: This sheaf is called **the dual vector bundle**, denoted by B^*. Its fibers are dual to the fibers of B.

DEFINITION: **Bilinear form** on a bundle $\mathcal{B}$ is a section of $(\mathcal{B} \otimes \mathcal{B})^*$. A symmetric bilinear form on a real bundle $\mathcal{B}$ is called **positive definite** if it gives a positive definite form on all fibers of $\mathcal{B}$. Symmetric positive definite form is also called **a metric**. A skew-symmetric bilinear form on $\mathcal{B}$ is called **non-degenerate** if it is non-degenerate on all fibers of $\mathcal{B}$.

Subbundles

DEFINITION: A subbundle $\mathcal{B}_1 \subset \mathcal{B}$ is a subsheaf of modules which is also a vector bundle.

DEFINITION: Direct sum $\oplus$ of vector bundles is a direct sum of corresponding sheaves.

EXAMPLE: Let $\mathcal{B}$ be a vector bundle equipped with a metric (that is, a positive definite symmetric form), and $\mathcal{B}_1 \subset \mathcal{B}$ a subbundle. Consider a subset $\operatorname{Tot}\mathcal{B}_1^\perp \subset \operatorname{Tot}\mathcal{B}$, consisting of all $v \in \mathcal{B}|_x$ orthogonal to $\mathcal{B}_1|_x \subset \mathcal{B}|_x$. **Then $\operatorname{Tot}\mathcal{B}_1^\perp$ is a total space of a subbundle, denoted as $\mathcal{B}_1^\perp \subset \mathcal{B}$,** and we have an isomorphism $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_1^\perp$.

REMARK: A total space of a direct sum of vector bundles $\mathcal{B} \oplus \mathcal{B}'$ **is homeomorphic to $\operatorname{Tot}\mathcal{B} \times_M \operatorname{Tot}\mathcal{B}'$.**

EXERCISE: Let $\mathcal{B}$ be a real vector bundle. **Prove that $\mathcal{B}$ admits a metric.**