Geometry of manifolds

Lecture 9: Serre-Swan theorem

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Locally trivial fibrations

DEFINITION: A smooth map $f: X \longrightarrow Y$ is called **a locally trivial fibration** if each point $y \in Y$ has a neighbourhood $U \ni y$ such that $f^{-1}(U)$ is diffeomorphic to $U \times F$, and the map $f: f^{-1}(U) = U \times F \longrightarrow U$ is a projection. In such situation, F is called **the fiber** of a locally trivial fibration.

DEFINITION: **A trivial fibration** is a map $X \times Y \longrightarrow Y$.

DEFINITION: **A total space of a vector bundle** on Y is a locally trivial fibration $f: X \longrightarrow Y$ with fiber $\mathbb{R}^n$, with each fiber $V := f^{-1}(y)$ equipped with a structure of a vector space, smoothly depending on $y \in Y$.

DEFINITION: **A vector bundle** is a locally free sheaf of $C^\infty M$-modules.

REMARK: Let $\pi: B \longrightarrow M$ be a total space of a vector bundle, $U \subset M$ open subset, and $\mathcal{B}(U)$ the space of all smooth sections of $\pi^{-1}(U) \xrightarrow{\pi} U$. **Then $\mathcal{B}$ is a locally free sheaf of $C^\infty M$-modules**.

REMARK: **This construction is an "equivalence of categories";** see below for a definition.

Categories

DEFINITION: A category $\mathcal{C}$ is a collection of data called "objects" and "morphisms between objects" which satisfies the axioms below.

DATA.

Objects: The set $\mathcal{O}\mathrm{b}(\mathcal{C})$ of **objects** of $\mathcal{C}$.

Morphisms: For each $X, Y \in \mathcal{O}\mathrm{b}(\mathcal{C})$, one has a set $\mathcal{M}\mathrm{or}(X, Y)$ of **morphisms from X to Y.**

Composition of morphisms: For each $\varphi \in \mathcal{M}\mathrm{or}(X, Y), \psi \in \mathcal{M}\mathrm{or}(Y, Z)$ there exists **the composition** $\varphi \circ \psi \in \mathcal{M}\mathrm{or}(X, Z)$

Identity morphism: For each $A \in \mathcal{O}\mathrm{b}(\mathcal{C})$ there exists a morphism $\mathrm{Id}_A \in \mathcal{M}\mathrm{or}(A, A)$.

AXIOMS.

Associativity of composition: $\varphi_1 \circ (\varphi_2 \circ \varphi_3) = (\varphi_1 \circ \varphi_2) \circ \varphi_3$.

Properties of identity morphism: For each $\varphi \in \mathcal{M}\mathrm{or}(X, Y)$, one has $\mathrm{Id}_x \circ \varphi = \varphi = \varphi \circ \mathrm{Id}_Y$.

Functors and equivalence of categories

DEFINITION: Let $\mathcal{C}_1, \mathcal{C}_2$ be categories. **A covariant functor** from $\mathcal{C}_1$ to $\mathcal{C}_2$ is the following collection of data.

(i) A map $F: \mathcal{O}b(\mathcal{C}_1) \longrightarrow \mathcal{O}b(\mathcal{C}_2)$.

(ii) A map $F: \mathcal{M}or(X,Y) \longrightarrow \mathcal{M}or(F(X),F(Y))$,

defined for each $X,Y \in \mathcal{O}b(\mathcal{C}_1)$.

These data define **a functor from $\mathcal{C}_1$ to $\mathcal{C}_2$**, if $F(\varphi)\circ F(\psi) = F(\varphi\circ\psi)$, and $F(\operatorname{Id}_X) = \operatorname{Id}_{F(X)}$.

DEFINITION: Two functors $F,G: \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ are called **equivalent** if for each $X \in \mathcal{O}b(\mathcal{C}_1)$ there exists an isomorphism $\Psi_X: F(X) \longrightarrow G(X)$, such that for each $\varphi \in \mathcal{M}or(X,Y)$ one has $F(\varphi)\circ\Psi_Y = \Psi_X \circ G(\varphi)$.

DEFINITION: A functor $F: \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is called **equivalence of categories** if there exist functors $G, G': \mathcal{C}_2 \longrightarrow \mathcal{C}_1$ such that $F\circ G$ is equivalent to an identity functor on $\mathcal{C}_1$, and $G'\circ F$ is equivalent to identity functor on $\mathcal{C}_2$.

EXAMPLE: Let $\mathcal{C}$ be a category of finite-dimensional vector spaces ovet $\mathbb{R}$ with a fixed basis (morphisms are linear maps), and $\mathcal{C}'$ a category with $\mathcal{O}b(\mathcal{C}') = \{\emptyset, \mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, ...\}$, and morphisms also linear maps. Prove that the inclusion map $\mathcal{C}' \longrightarrow \mathcal{C}$ **is an equivalence of categories,** but **not an isomorphism**.

Total space of a vector bundle from its sheaf of sections

DEFINITION: Category of vector bundles $\mathcal{C}_b$ is a category where objects are locally free $C^\infty M$-sheaves, and morphisms are morphisms of $C^\infty M$-sheaves such that all kernels and cokernels are locally free.

EXERCISE: Prove that it is a category.

DEFINITION: Category of total spaces of vector bundles $\mathcal{C}_t$ is a category where objects are total spaces of vector bundles, and morphisms of total spaces over M are maps $B_1 \longrightarrow B_2$ compatible with projection to M, the multiplicative structure, and of constant rank at each fiber.

EXERCISE: Prove that it is a category.

THEOREM: Let $\pi: B \longrightarrow M$ be a total space of a vector bundle, $U \subset M$ open subset, and $\mathcal{B}(U)$ the space of all smooth sections of $\pi^{-1}(U) \xrightarrow{\pi} U$. Then this map **defines an equivalence of categories** $\mathcal{C}_b \xrightarrow{\sim} \mathcal{C}_t$.

REMARK: The proof was given in the last lecture, using different language.

EXERCISE: Produce a proof of this theorem.

Tensor product

DEFINITION: Let V, V' be R-modules, W a free abelian group generated by $v \otimes v'$, with $v \in V, v' \in V'$, and $W_1 \subset W$ a subgroup generated by combinations $rv \otimes v' - v \otimes rv'$, $(v_1 + v_2) \otimes v' - v_1 \otimes v' - v_2 \otimes v'$ and $v \otimes (v'_1 + v'_2) - v \otimes v'_1 - v \otimes v'_2$. Define **the tensor product** $V \otimes_R V'$ as a quotient group W/W_1.

EXERCISE: Show that $r \cdot v \otimes v' \mapsto (rv) \otimes v'$ **defines an R-module structure on $V \otimes_R V'$.**

REMARK: Let $\mathcal{F}$ be a sheaf of rings, and $\mathcal{B}_1$ and $\mathcal{B}_2$ be sheaves of locally free $(M, \mathcal{F})$-modules. **Then**

$$U \longrightarrow \mathcal{B}_1(U) \otimes_{\mathcal{F}(U)} \mathcal{B}_2(U)$$

is also a locally free sheaf of modules.

DEFINITION: Tensor product of vector bundles is a tensor product of the corresponding sheaves of modules.

Dual bundle and bilinear forms

DEFINITION: Let V be an R-module. **A dual R-module** V^* is $\operatorname{Hom}_R(V, R)$ with the R-module structure defined as follows: $r \cdot h(\dots) \mapsto rh(\dots)$.

CLAIM: Let $\mathcal{B}$ be a vector bundle, that is, a locally free sheaf of $C^\infty M$-modules, and $\operatorname{Tot}\mathcal{B} \stackrel{\pi}{\longrightarrow} M$ its total space. Define $\mathcal{B}^*(U)$ as a space of smooth functions on $\pi^{-1}(U)$ linear in the fibers of π. **Then $\mathcal{B}^*(U)$ is a locally free sheaf over $C^\infty(M)$.**

DEFINITION: This sheaf is called **the dual vector bundle**, denoted by B^*. Its fibers are dual to the fibers of B.

DEFINITION: **Bilinear form** on a bundle $\mathcal{B}$ is a section of $(\mathcal{B} \otimes \mathcal{B})^*$. A symmetric bilinear form on a real bundle $\mathcal{B}$ is called **positive definite** if it gives a positive definite form on all fibers of $\mathcal{B}$. Symmetric positive definite form is also called **a metric**. A skew-symmetric bilinear form on $\mathcal{B}$ is called **non-degenerate** if it is non-degenerate on all fibers of $\mathcal{B}$.

Subbundles

DEFINITION: A subbundle $\mathcal{B}_1 \subset \mathcal{B}$ is a subsheaf of modules which is also a vector bundle, and such that the quotient $\mathcal{B}/\mathcal{B}_1$ is also a vector bundle.

DEFINITION: Direct sum $\oplus$ of vector bundles is a direct sum of corresponding sheaves.

EXAMPLE: Let $\mathcal{B}$ be a vector bundle equipped with a metric (that is, a positive definite symmetric form), and $\mathcal{B}_1 \subset \mathcal{B}$ a subbundle. Consider a subset $\operatorname{Tot}\mathcal{B}_1^\perp \subset \operatorname{Tot}\mathcal{B}$, consisting of all $v \in \mathcal{B}|_x$ orthogonal to $\mathcal{B}_1|_x \subset \mathcal{B}|_x$. **Then $\operatorname{Tot}\mathcal{B}_1^\perp$ is a total space of a subbundle, denoted as $\mathcal{B}_1^\perp \subset \mathcal{B}$,** and we have an isomorphism $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_1^\perp$.

REMARK: A total space of a direct sum of vector bundles $\mathcal{B} \oplus \mathcal{B}'$ **is homeomorphic to $\operatorname{Tot}\mathcal{B} \times_M \operatorname{Tot}\mathcal{B}'$.**

EXERCISE: Let $\mathcal{B}$ be a real vector bundle. **Prove that $\mathcal{B}$ admits a metric.**

PROPOSITION: Let $A \subset B$ be a sub-bundle. **Then $B \cong A \oplus C$.**

Proof: Find a positive definite metric on B, and set $C := B^\perp$. ■

Tangent bundle

PROPOSITION: Let $M \subset \mathbb{R}^n$ be a smooth submanifold of $\mathbb{R}^n$, and $TM \subset \mathbb{R}^n \times \mathbb{R}^n$ the set of all pairs $(v, x) \in M \times \mathbb{R}^n$, where $x \in M \times \mathbb{R}^n$ is a point of M, and $v \in \mathbb{R}^n$ a vector tangent to M in m, that is, satisfying

$$\lim_{t \longrightarrow 0} \frac{d(M, m+tv)}{t} \longrightarrow 0.$$

Then the natural additive operation on $TM \subset M \times \mathbb{R}^n$ (addition of the second argument) and a multiplication by real numbers **defines on TM a structure of a relative vector space over M**, that is, makes TM a total space of a vector bundle. Moreover, this vector bundle is isomorphic to a tangent bundle, that is, to the sheaf $\operatorname{Der}_{\mathbb{R}}(C^\infty M)$.

Proof. Step 1: For each $z \in M$, we can choose coordinates in a neighbourhood of z in $\mathbb{R}^n$ in such a way that $M = \mathbb{R}^k \subset \mathbb{R}^n$. Therefore, **it would suffice to prove proposition when $M = \mathbb{R}^k \subset \mathbb{R}^n$.**

Proof. Step 2: In this case, $TM = \mathbb{R}^k \times \mathbb{R}^k$ **is a total space of a vector bundle,** of the same dimension as the tangent bundle. It remains to construct a sheaf morphism from the sheaf of sections of TM to $\operatorname{Der}_{\mathbb{R}}(C^\infty M)$, inducing an isomorphism.

Tangent bundle (cont.)

Proof. Step 3: Let $\pi_x : \mathbb{R}^n \longrightarrow T_xM$ be an orthogonal projection map. By the inverse function theorem, **$\pi_x|_M : M \longrightarrow T_xM$ is a diffeomorphism in a neighbourhood of $x \in M$.** Let $U_x \subset T_xM$ be such an open neighbourhood and $\pi_x^{-1}(U_x) \xrightarrow{\pi_x} U_x$ a diffeomphism.

Proof. Step 4: For each vector $v \in T_xM$, and $f \in C^\infty M$, let $D_v(f)$ be the derivative of $\tilde{f} \in C^\infty U_x$ along v, where $\tilde{f}(z) = f(\pi_x^{-1}(z))$. Then a section $\gamma \in TM(U)$ defines a derivation $D_\gamma(f)(z) := D_{\gamma|_z}(f)$. **We obtained a sheaf homomorphism $TM \xrightarrow{\Psi} \operatorname{Der}_{\mathbb{R}}(C^\infty M)$.**

Proof. Step 5: The vector bundles TM and $\operatorname{Der}_{\mathbb{R}}(C^\infty M)$ have the same dimension, and for each non-zero vector $v \in T_xM$, **the corresponding derivation is non-zero, hence $\ker \Psi = 0$.** ■

DEFINITION: The tangent bundle of M, as well as its total space, is denoted by TM. When one wants to distinguish the total space and the tangent bundle, one writes $\operatorname{Tot}(TM)$.

Pullback

CLAIM: Let $M_1 \xrightarrow{\varphi} M$ be a smooth map of manifolds, and $B \xrightarrow{\pi} M$ a total space of a vector bundle. **Then $B \times_M M_1$ is a total space of a vector bundle on M_1.**

Proof. Step 1: $B \times_M M_1$ is obviously a relative vector space. Indeed, the fibers of projection $\pi_1 : B \times_M M_1 \longrightarrow M_1$ are vector spaces, $\pi_1^{-1}(m_1) = \pi^{-1}(\varphi(m_1))$. It remains only to show that it is locally trivial.

Step 2: Consider an open set $U \subset M$ that $B|_U = U \times \mathbb{R}^n$, and let $U_1 := \varphi^{-1}U$. Then $B \times_U U_1 = U_1 \times \mathbb{R}^n$. **Since M_1 is covered by such U_1, this implies that π_1 is a locally trivial fibration,** and the additive structure smoothly depends on $m_1 \in M_1$. ■

DEFINITION: The bundle $\pi_1 : B \times_M M_1 \longrightarrow M_1$ is denoted $\varphi^* B$, and called **inverse image**, or **a pullback** of B.

Pullback and the tangent bundle

CLAIM: Let $j: M \hookrightarrow N$ be a closed embedding of smooth bundles. **Then there is a natural injective morphism of vector bundles $TM \hookrightarrow j^* TN$.**

Proof: Using Whitney's theorem, we embed N to $\mathbb{R}^n$. Then $j^*TN \subset M \times \mathbb{R}^n$ is the set of pairs $x \in M, v \in T_x N$. **The bundle TM is embedded to j^*TN, because each tangent vector to M is also tangent to N.** ■

EXERCISE: Prove that the map $TM \hookrightarrow j^*TN$ **is independent from the choice of embedding $N \subset \mathbb{R}^n$.**

COROLLARY: Let M be a manifold, and $j: M \hookrightarrow \mathbb{R}^n$ a closed embedding. **Then TM is a direct summand of a trivial bundle $j^*T\mathbb{R}^n$.**

Any bundle is a direct summand of a trivial bundle

THEOREM: Any vector bundle on a metrizable manifold is a direct summand of a trivial bundle.

Proof. Step 1: Let B be a vector bundle on M, and $\operatorname{Tot} B$ its total space. Consider the tangent bundle $T \operatorname{Tot} B$, and let $M \stackrel{\varphi}{\hookrightarrow} \operatorname{Tot} B$ be an embedding corresponding to a zero section. **Then the pullback $\varphi^* T \operatorname{Tot} B$ is isomorphic (as a bundle) to the direct sum $TM \oplus B$.**

Step 2: Using Whitney's theorem, find a closed embedding $j: \operatorname{Tot} B \longrightarrow \mathbb{R}^n$. **This gives injective morphisms of vector bundles**

$$B \hookrightarrow TM \oplus B = \varphi^*(T \operatorname{Tot} B) \hookrightarrow (\varphi j)^* T\mathbb{R}^n.$$

However, $(\varphi j)^* T\mathbb{R}^n$ is trivial, because the bundle $T\mathbb{R}^n$ is trivial. ■

Projective modules

DEFINITION: Let V be an R-module, and $V' \subset V$ its submodule. Assume that V contains a submodule V'', not intersecting V', such that V' together with V'' generate V. In this case, V' and V'' are called **direct summands** of V, and V – **a direct sum** of V' and V''. This is denoted $V = V' \oplus V''$.

DEFINITION: An R-module is called **projective** if it is a direct summand of a free module $\bigoplus_I R$ (possibly of infinite rank).

COROLLARY: Let $\mathcal{B}$ be a vector bundle, and B its space of sections, considered as a $C^\infty M$-module. **Then B is projective.**

THEOREM: (Serre-Swan theorem) Let $\mathcal{C}_p$ be a category with objects projective $C^\infty M$-modules, and morphisms homomorphism of $C^\infty M$-modules with kernels and cokernels projective, $\mathcal{C}_b$ the category of vector bundles, and $\Psi: \mathcal{C}_b \longrightarrow \mathcal{C}_b$ a functor mapping B to its space of global sections. **Then Ψ is an equivalence of categories.**

Proof later.

Determinant bundle

DEFINITION: A line bundle is a 1-dimensional vector bundle.

EXERCISE: Let M be a simply connected manifold. **Prove that any real line bundle on M is trivial.**

DEFINITION: Let B be a vector bundle of rank n, and $\Lambda^n B$ its top exterior product. This bundle is called **determinant bundle** of B.

REMARK: It is a line bundle.

REMARK: Let M be an n-manifold, and $\Lambda^n TM$ a determinant bundle of its tangent bundle. Prove that $\Lambda^n TM$ **is trivial if and only if M is orientable.**

DEFINITION: A real vector bundle is called **orientable** if its determinant bundle is trivial.

Trivializations and determinant

DEFINITION: Recall that **a trivialization** of a vector bundle B over U is a set of **free generators** of B, that is, sections $x_1, ..., x_n \in B$ such that the map $\nu: \ (C^\infty U)^n \longrightarrow B|_U$ mapping generators $e_i \in (C^\infty U)^n$ to x_i is an isomorphism.

DEFINITION: Let $x \in M$ be a point on a manifold. Denote by $\mathfrak{m}_x \subset C^\infty M$ the ideal of all functions vanishing in x. Let $\mathcal{B}$ be a sheaf of $C^\infty M$-modules, and b a section of $\mathcal{B}$. We say that b **nowhere vanishes** on $U \subset M$ if its germ b_x does not lie in $\mathfrak{m}_x\mathcal{B}$ for each $x \in U$.

PROPOSITION: Let B be a vector bundle, and $x_1, ..., x_n \in B$ be a set of sections which are linearly independent in $B/\mathfrak{m}_{z_0}B$ and generate $B/\mathfrak{m}_{z_0}B$, for a fixed point $z_0 \in M$. Let $\xi \in \Lambda^n B$, $\xi := x_1 \wedge x_2 \wedge ... \wedge x_n$ be the determinant of x_i, considered as a section of a line bundle $\det B$. Suppose that ξ nowhere vanishes on $U \subset M$. **Then $\{x_i|_U\}$ are free generators of $B|_U$.**

Proof: Define a map $\nu: \ (C^\infty U)^n \longrightarrow B|_U$ mapping generators $e_i \in (C^\infty U)^n$ to x_i. **This map induces an isomorphism on each fiber, hence bijective.** The inverse function theorem implies that it is a diffeomorphism. ■

A stalk of a $C^\infty M$-module

DEFINITION: Let $x \in M$ be a point on a manifold. **A stalk** of a $C^\infty M$-module V is a tensor product $C^\infty_x M \otimes_{C^\infty M} V$, where $C^\infty_x M$ is a ring of germs of $C^\infty M$ in x. We consider a stalk V_x as a $C^\infty_x M$-module.

REMARK: Let V be a free $C^\infty M$-module. Then stalk of the space of sections $V(M)$ in x is a stalk of the sheaf V in x.

CLAIM: Let A be a free $C^\infty M$-module of rank n, decomposed as a direct sum of two projective modules: $A = B \oplus C$. We identify A with a space of sections of a trivial sheaf of $C^\infty M$-modules, denoted by $\mathcal{A}$. **Let $\mathcal{B} \subset \mathcal{A}$ be a subsheaf consisting of all sections $\gamma \in \mathcal{V}(U)$, such that the germs of γ at each $x \in M$ lie in the stalk B_x.** Define $\mathcal{C} \subset \mathcal{A}$ in a similar fashion. Then

(i) $\mathcal{B}$, $\mathcal{C}$ are sheaves of $C^\infty M$-modules.

(ii) $\mathcal{A} = \mathcal{B} \oplus \mathcal{C}$.

(iii) **The sheaves $\mathcal{B}$, $\mathcal{C}$ are locally free.**

Proof: Next slide.

The proof of Serre-Swan theorem

CLAIM: Let A be a free $C^\infty M$-module of rank n, decomposed as a direct sum of two projective modules: $A = B \oplus C$. We identify A with a space of sections of a trivial sheaf of $C^\infty M$-modules, denoted by $\mathcal{A}$. **Let $\mathcal{B} \subset \mathcal{A}$ be a subsheaf consisting of all sections $\gamma \in \mathcal{V}(U)$, such that the germs of γ at each $x \in M$ lie in the stalk B_x.** Define $\mathcal{C} \subset \mathcal{A}$ in a similar fashion. Then

(i) $\mathcal{B}$, $\mathcal{C}$ are sheaves of $C^\infty M$-modules.

(ii) $\mathcal{A} = \mathcal{B} \oplus \mathcal{C}$.

(iii) **The sheaves $\mathcal{B}$, $\mathcal{C}$ are locally free.**

Proof: The first two claims are clear.

Fix $z \in M$. Let $x_1, ..., x_k$ be sections of $\mathcal{B}$ generating $\mathcal{B}/\mathfrak{m}_z\mathcal{B}$ and $y_1, ..., y_l$ sections of $\mathcal{C}$ generating $\mathcal{C}/\mathfrak{m}_z\mathcal{C}$. Choose them to be linearly independent, and let U be an open neighbourhood of z such that the section $x_1 \wedge x_2 \wedge ... \wedge x_k \wedge y_1 \wedge ... \wedge y_l \in \Lambda^n B$ is nowhere degenerate on U. Then $\{x_i, y_j\}$ are free generators of $\mathcal{A}$, hence $\{x_i\}$ are free generators of $\mathcal{B}$ and $\{y_j\}$ are free generators of $\mathcal{C}$. **We have shown that these sheaves are locally free**. ■

REMARK: This gives a way of reconstructing a vector bundle from a projective $C^\infty M$-module. **The rest of the proof of Serre-Swan is left as an exercise.**