# Test assignment #1

**Rules:** Please solve this in class, before 21:30 February 04, 2013, and give me the written solutions. The score for the test is computed using the formula $s = 2p - (\max(p-4, 0))$ (rounded down). Results will be announced at `http://bogomolov-lab.ru/KURSY/GEOM-2013/`

**Exercise 1.1.** Find a smooth map $\mathbb{R} \longrightarrow S^3$ with dense image, or prove that it does not exist.

**Exercise 1.2.** Find a smooth map $[0,1] \longrightarrow S^3$ with dense image, or prove that it does not exist.

**Exercise 1.3.** Let $Z \subset M$ be a dense subset in a manifold, and $U \supset Z$ an open subset of $M$ containing $Z$. Show that $U = M$, or find a counterexample.

**Exercise 1.4.** Let $M$ be a countable, connected metric space. Show that $M$ is never infinite.

**Exercise 1.5 (2 points).** Let $M$ be a compact Hausdorff space, $R$ – ring of continuous functions on $M$ with values in $\mathbb{Z}/2\mathbb{Z}$, and $I \subset R$ a maximal ideal. Prove that there exists a point $x \in M$ such that all functions $f \in I$ vanish at $x$.

**Exercise 1.6.** A continuous map $f: X \longrightarrow Y$ of topological spaces is called **proper** if a preimage of any compact set is compact, **closed** if an image of any closed set is closed, and **open** if an image of any open set is open. Find an example of a continuous map $f: X \longrightarrow Y$ of Hausdorff topological spaces which is

a. open, not proper, not closed

b. (2 points) closed, not proper, not open

c. (2 points) proper, not open, not closed

or show that it does not exist.

**Exercise 1.7.** Let $M := \mathbb{R}^2 \backslash \mathbb{Q}^2$. Prove that $M$ is connected.

**Exercise 1.8.** Let $X = \mathbb{R}P^n$, and $Y = (S^1)^m$. Show that any continous map $f: X \longrightarrow Y$ is homotopic to a trivial one.

**Exercise 1.9.** Let $Z \subset \mathbb{R}^n$ be a countable set. Construct a function $\mu: \mathbb{R}^n \longrightarrow \mathbb{R}$ which is continuous at $x \notin Z$ and discontinuous at $Z$.

**Exercise 1.10.** Let $f_i: [0,1] \longrightarrow [0,1]$ be a sequence of continuous functions, and $f(z) := \lim_i f_i(z)$. Prove that $f$ is continuous, and find a counterexample.