Hyperkähler manifolds, exam

Each student receives a random selection of 15 test problems (the output of the randomizer is printed on a separate sheet). You are expected to be able to prove all theorems you use, unless stated otherwise. You are free to use the Riemann-Roch formula, provided you can state it correctly. The final score is given by $s = b + 3$, where b is the sum of points you got.

1 Holonomy groups and connections

Exercise 1.1. Let $V = \mathbb{C}^{2n}$ be the tautological representation of $\operatorname{Sp}(n)$. Prove that $\Lambda^{2k+1}(V)$ has no $\operatorname{Sp}(n)$-invariant vectors.

Exercise 1.2. Let $V = \mathbb{C}^{2n}$ be the tautological representation of $\operatorname{Sp}(n)$. Prove that any $\operatorname{Sp}(n)$-invariant vector in $\Lambda^2(V)$ is proportional to the symplectic form.

Exercise 1.3. Let $V = \mathbb{C}^{2n}$ be the tautological representation of $\operatorname{Sp}(n)$. Prove that $\Lambda^3(V)$ is an irreducible representation of $\operatorname{Sp}(n)$, or find counterexamples.

Exercise 1.4. Let M be a hyperkähler manifold with global holonomy $\operatorname{Sp}(n_1) \times \operatorname{Sp}(n_2) \times ... \times \operatorname{Sp}(n_k)$, with $2\sum_{i=1}^k n_i = \dim_{\mathbb{C}} M$. Prove that M is simply connected.

Exercise 1.5 (3 points). Let (M, ω) be a symplectic manifold. Prove that M admits a torsion-free connection ∇ such that $\nabla(\omega) = 0$.

Exercise 1.6 (2 points). Find an example of a compact n-manifold M not admitting a Riemannian metric of signature $(1, n-1)$.

2 Hyperkähler manifolds

Remark 2.1. In this section you can use the Fujiki formula, provided you can state it correctly.

Definition 2.1. An algebraic fuction $A : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a continuous function which is given as one of the branches of a multi-valued function μ with the graph $\Gamma_\mu \subset \mathbb{R}^n \times \mathbb{R}$ which is an irreducible algebraic variety.

Exercise 2.1 (2 points). Let $A: \mathbb{R}^n \longrightarrow \mathbb{R}$ be an algebraic function, and $S \subset \mathsf{Gr}(k,n)$ an open subset in the Grassmanian, with $k > 1$. Assume that A is polynomial on each $l \in S$. Prove that A is polynomial.

Exercise 2.2 (2 points). Let M be a hyperkähler manifold of maximal holonomy, and $M \longrightarrow B$ a surjective holomorphic map to a Kähler manifold B. Prove that $\dim B = 0, n, 2n$, where $2n = \dim_{\mathbb{C}} M$.

Exercise 2.3 (3 points). Let $L \subset M$ be a complex Lagrangian submanifold in a compact hyperkähler manifold. Prove that L is projective.

Exercise 2.4. Let M be a hyperkähler manifold of maximal holonomy, and $M \longrightarrow B$ a surjective holomorphic map to a Kähler manifold B, $\dim B < \dim M$. Prove that B is projective.

Exercise 2.5. Let $\eta \in H^2(M)$ be a non-zero class on a hyperkähler manifold of maximal holonomy, $\dim_{\mathbb{C}} M = 2n$. Prove that $\eta^n \neq 0$.

3 Trianalytic subvarieties

Exercise 3.1. Find a compact hypercomplex manifold (M, I, J, K) such that for any induced complex structure $L = aI + bJ + cK$, the manifold (M, L) has non-trivial divisors.

Exercise 3.2. Let (M, I, J, K) be a hyperkähler manifold, and $\phi: (M, I) \longrightarrow (M, I)$ a holomorphic automorphism which acts trivially on $H^2(M)$. Prove that its graph is trianalytic.

Exercise 3.3 (3 points). Let T be a complex 2-dimensional torus, $\operatorname{Hilb}^n(T)$ its Hilbert scheme, and $\operatorname{Hilb}^n(T) \xrightarrow{a} T = \operatorname{Alb}(\operatorname{Hilb}^n(T))$ the Albanese map. Prove that the fundamental group of $a^{-1}(0)$ is finite.

Definition 3.1. We call the universal cover of $a^{-1}(0)$ **the generalized Kummer variety**.

Exercise 3.4 (2 points). Let ι be an involution of the torus mapping x to $-x$. Prove that it gives a holomorphic involution of the generalized Kummer variety, and its fixed point set is trianalytic.

Exercise 3.5 (2 points). Let (M, I, J, K) be a hyperkähler manifold, and $X \subset (M, I)$ a complex subvariety of dimension 2. Prove that $\int_X \omega_I^2 \geqslant \frac{1}{2}\int_x (\Omega \wedge \bar{\Omega})$. Prove that equality is realized if and only if X trianalytic.

Exercise 3.6 (2 points). Let $X \subset T^{4n}$ be a trianalytic subvariety in a compact torus. Prove that X is a union of subtori.

Exercise 3.7. Let X be a hyperkähler manifold (not necessarily compact) with $b_2(X) = 1$. Prove that there exists an induced complex structure L on X such that (X, L) contains no compact complex subvarieties.

4 Hypercomplex manifolds

Exercise 4.1. Let M be a compact hypercomplex manifold, $\dim_{\mathbb{R}} M = 4n$ and η a closed $SU(2)$-invariant 2-form. Suppose that η has maximal rank in some point $m \in M$. Prove that $\int_M \eta^{2n} \neq 0$.

Exercise 4.2. Let be a compact hypercomplex manifold, $\dim_{\mathbb{R}} M = 4n$ and V the bundle of $SU(2)$-invariant 2-forms. Prove that $\dim V = \dim \mathsf{Sym}^2_{\mathbb{C}} \Lambda^{1,0}(M, I)$.

Exercise 4.3. Let (M, I, K, K) be a compact hypercomplex manifold, and $\eta \in \Lambda^{2,0}(M, I)$ a 2-form which satisfies $\partial\eta = 0$, $J\eta = \bar{\eta}$. Prove that $d\eta$ has weight 1 with respect to $SU(2)$.

Exercise 4.4 (2 points). Let (M, I, K, K) be a compact hypercomplex manifold, and $V \subset \Lambda^{2,0}(M, I)$ a space of 2-forms which satisfy $\partial\eta = 0$, $J\eta = \bar{\eta}$. Prove that V is finitely dimensional, or find a counterexample.

Exercise 4.5 (2 points). Let M be a compact hypercomplex manifold, $\dim_{\mathbb{R}} M = 4n$ and V the space of closed $SU(2)$-invariant 2-forms. Prove that V is finitely-dimensional.

Exercise 4.6. Let M be a compact hypercomplex manifold, and η the curvature of the Obata connection of the canonical bundle of (M, I).

a. Prove that η is $SU(2)$-invariant.

b. (2 points) Prove that $\eta^{2n} = 0$, where $\dim_{\mathbb{R}} M = 4n$.

5 Stable bundles

Exercise 5.1. Let B be a flat Hermitian bundle over a Kähler manifold, and $B_1 \subset B$ a holomorphic sub-bundle, not necessarily flat. Denote the curvature of the Chern connection on B_1 by Θ_{B_1}. Prove that $\sqrt{-1}\operatorname{Tr}_B \Lambda(\Theta_{B_1}) \geqslant 0$, and equality happens if and only if the Chern connection of B preserves B_1.

Exercise 5.2. Let (G, I) be a compact Lie group with a left-invariant complex structure. Prove that (G, I) admits a holomorphic, G-invariant map to a homogeneous projective manifold. Prove that the canonical bundle of (G, I) has no non-zero holomorphic sections, unless G is a compact torus.

Exercise 5.3. Let F be a stable coherent sheaf on a smooth Kähler manifold M. Prove that $F^{**} := \operatorname{Hom}(\operatorname{Hom}(F, \mathcal{O}_M), \mathcal{O}_M)$ is stable.

Exercise 5.4. Let X be a compact complex manifold equipped with a free action of the cyclic group G, and $M := X/G$.

a. (2 points) Suppose that all stable coherent sheaves on M have rank 1. Prove that all stable sheaves on X have rank 1.

b. Suppose that all stable coherent sheaves on X have rank 1. Prove that all stable sheaves on M have rank 1.

(don't use Donaldson-Uhlenbeck-Yau unless you can prove it).

Definition 5.1. A vector bundle B with $c_1 B = 0$ is called **Bogomolov stable** when all sections $B^{\otimes n} \otimes (B^*)^{\otimes M}$ are sections of rank one subsheaves which are direct summands.

Exercise 5.5. Using Donaldson-Uhlenbeck-Yau, prove that μ-stability implies Bogomolov stability.

Exercise 5.6 (2 points). Let B be a Bogomolov stable bundle on $\mathbb{C}P^n$, with $c_1(B) = 0$. Prove that B is μ-stable.

Exercise 5.7 (2 points). Prove that the tangent bundle $T\mathbb{C}P^2$ is stable (without using Donaldson-Uhlenbeck-Yau).