Class assignment 1: quaternionic Hermitian structures

Exercise 1.1. Let ∇ be a torsion-free connection on a manifold M, and ω a differential form which satisfies $\nabla(\omega) = 0$. Prove that ω is closed.

Exercise 1.2. Let ∇ be a torsion-free connection on a manifold M, and I an almost complex structure such that $\nabla(I) = 0$. Prove that I is integrable.

Exercise 1.3. Let M be an even-dimensional smooth manifold, A the (infinitely-dimensional) space of all non-degenerate 2-forms on M, and B the space of all almost complex structures on M. Prove that A and B are homotopy equivalent.

Definition 1.1. An almost hypercomplex structure on a manifold M is a triple almost complex structures (I, J, K) satisfying the quaternionic relations. **An almost hypercomplex Hermitian structure** on M is an almost complex structure (I, J, K) and a Riemannian metric h which is invariant under the action of I, J, K.

Exercise 1.4. Let (M, I) be almost complex manifold, A the (infinitely-dimensional) space of all non-degenerate (2,0) forms, and B the space of all almost hypercomplex Hermitian structures (I, J, K, h). Prove that A and B are homotopy equivalent.

Exercise 1.5. Let (M, I, J, K) be a hypercomplex Hermitian manifold, and $\omega_I, \omega_J, \omega_K$ its fundamental forms. Suppose that these forms are closed. Prove that (M, I, J, K) is hyperkähler.

Exercise 1.6. Let (M, I, J, K) be a hyperkähler manifold, $a, b, c, a^2 + b^2 + c^2$ real numbers, and $L = aI+bJ+cK$ the corresponding almost complex structure. Prove that L is integrable.

Exercise 1.7 (*). Let $\omega_1, \omega_2, \omega_3$ be a triple of 2-forms on a manifold M such that any non-zero linear combination of ω_i is non-degenerate.

a. Prove that there exists a hypercomplex Hermitian structure with fundamental forms $\omega_I, \omega_J, \omega_K$ such that the 3-dimensional bundles spanned by $\omega_I, \omega_J, \omega_K$ and $\omega_1, \omega_2, \omega_3\rangle$ coincide, or find a counterexample.

b. Suppose that all ω_i are closed. Prove that there exists a torsion-free connection preserving ω_i, or find a counterexample.

Exercise 1.8 (*). Let $\omega_1, \omega_2, \omega_3$ be a triple of 2-forms on a manifold M such that any non-zero linear combination of ω_i is non-degenerate or has constant rank $\frac{1}{2} \dim M$, but not always non-degenerate. Prove that TM is equipped with an action of the matrix algebra $\mathrm{Mat}(2, \mathbb{R})$ preserving $\langle\omega_1, \omega_2, \omega_3\rangle$.