Class assignment 2: almost complex structures

Exercise 2.1. Construct a left-invariant, integrable almost complex structure on the Lie group $SU(2) \times SU(2)$.

Exercise 2.2. Let M be a compact complex surface.

a. Prove that all holomorphic differential forms on M are closed.

b. Find a compact complex 3-manifold which admits a non-closed holomorphic differential form.

Definition 2.1. Let M be an n-dimensional complex manifold. Fixing a volume form, we may identify $\Lambda^{1,1}(TM)$ (pseudo-Hermitian forms on T^*M and $\Lambda^{n-1,n-1}(M)$. A form $\eta \in \Lambda^{n-1,n-1}(M)$ is called **(strictly) positive** if all eigenvalues of the corresponding (1,1)-form on T^*M are (strictly) positive.

Exercise 2.3. a. Prove that any strictly positive $(n-1, n-1)$-form η is equal to ω^{n-1} for some Hermitian form ω on M.

b. Find a counterexample when η is not necessarily positive.

Exercise 2.4. Let (M, I) be a smooth almost complex manifold equipped with a transitive action of a group G. Assume that I is G-invariant (such a manifold is called **homogeneous**). Assume, moreover, that for some $x \in M$ there exists an element $\tau_x \in G$ fixing x. Consider the induced action of τ_x on T_xM; denote this operator by τ.

a. Suppose that $\tau = \lambda \operatorname{Id}$, where $\lambda \in \mathbb{R}$. Prove that for all $\lambda \neq 1$, the almost complex structure I is integrable.

b. Construct examples of such (M, I), G and τ_x for each $\lambda \in \mathbb{R}$.

c. Construct a homogeneous almost complex manifold which is not integrable.

d. Suppose that τ is not a scalar, but all its eigenvalues α_i satisfy $9 < |\alpha_i| < 10$. Prove that the almost complex structure I is integrable.