Class assignment 3: spinors and Clifford algebras

Definition 3.1. The Clifford algebra of a vector space V with a scalar product q is an algebra generated by V with a relation $xy + yx = -2q(x,y)1$.

Exercise 3.1. Let $A = A_{\mathsf{even}} \oplus A_{\mathsf{odd}}$ be a graded associative algebra. Let $A^\perp$ be the same vector space with new multiplication $a \bullet a' := (-1)^{\tilde{a}\tilde{a}'} aa'$. Prove that $\operatorname{Cl}(V,g)^\perp = \operatorname{Cl}(V,-g)$.

Definition 3.2. Let (V,g) be an oriented real vector space with orthogonal basis $e_1, ..., e_n$ such that $g(e_i,e_i) = \pm 1$. **A unit pseudoscalar** in $\operatorname{Cl}(V,g)$ is $\varepsilon := e_1 e_2 e_3 ... e_n$.

Exercise 3.2. Prove that the pseudoscalar is invariant with respect to the natural $SO(n)$-action. Prove that it is defined uniquely up to a sign.

Exercise 3.3. Prove the isomorphism $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} = \operatorname{Mat}(4,\mathbb{R})$.

Exercise 3.4. Let V be a vector space over a field of characteristic 0.

a. Prove that the automorphism group $\operatorname{Aut}(\operatorname{Mat}(V))$ is isomorphic to $PGL(V)$ (the quotient of $GL(V)$ by its center).

b. (*) Is it true for all fields?

Exercise 3.5. Prove that $\operatorname{Spin}(3,\mathbb{C}) \cong SL(2,\mathbb{C})$.

Definition 3.3. Define **the real spinor group** $\operatorname{Spin}(n,\mathbb{R})$ as a subgroup of the cover $\operatorname{Spin}(n,\mathbb{C}) \longrightarrow SO(n,\mathbb{C})$, fixed by the natural anticomplex involution, obtained from the standard anticomplex involution of $SO(n,\mathbb{C})$.

Exercise 3.6. Prove that this involution is always lifted to an involution of $\operatorname{Spin}(n,\mathbb{C})$.

Exercise 3.7. Prove that $\operatorname{Spin}(3,\mathbb{R}) \cong SU(2)$.

Exercise 3.8. Prove that $\operatorname{Spin}(n,\mathbb{R})$ is a non-trivial 2-sheeted cover of $SO(n)$ for all $n \geqslant 3$.