Class assignment 4: spinors and holonomy

Definition 4.1. Let U be a space equpped with scalar product. Consider **the exterior multiplication operator** $e_u : \Lambda^*(U) \longrightarrow \Lambda^{*+1}(U)$ with $e_u(\alpha) = u \wedge \alpha$ and **the convolution operator** $i_v : \Lambda^*(U) \longrightarrow \Lambda^{*-1}(U)$, with $i_v(\alpha)(v_1, ..., v_k) = \alpha(v, v_1, ..., v_k)$.

Exercise 4.1. Fix a basis u_i in U, and let v_j be the dual basis in U^*. For any pair of monomials A, B in $\Lambda^*(U)$, find a sequence $z_1, ..., z_r$, with each z_k equal to i_{v_i} or e_{u_j}, such that $z_1 z_2 ... z_r$ maps A to B and puts all other monomials to 0.

Exercise 4.2. Let (M, ∇) be a manifold with holonomy $\mathrm{Sp}(n)$. Prove that all parallel $(p, 0)$-forms on M are powers of the holomorphic symplectic form.

Exercise 4.3. Let (M, ∇) be a manifold with holonomy $SU(n)$. Prove that any holomorphic $(p, 0)$-form on M is a parallel section of the canonical bundle.

Definition 4.2. Let V be a vector space with non-degenerate scalar product g, and S a non-trivial, irreducible module over $\mathrm{Cl}(V)$ (that is, a representation of this algebra). Then S is called **the space of spinors** over V.

Exercise 4.4. Let V be a vector space with non-degenerate scalar product g, $v \in V$ a vector with $g(v, v) \neq 0$, and S the space of spinors. Prove that the Clifford multiplication $S \xrightarrow{\sigma(v)} S$ is invertible. Is the condition $g(v, v) \neq 0$ necessary?

Exercise 4.5. Let S be the space of spinors over V, and $\psi \in S$ a spinor. Consider the Clifford multiplication map $r_\psi : V \longrightarrow S$ mapping v to $\sigma(v)\psi$. Prove that $\dim \ker r_\psi \leqslant 1/2 \dim V$ when $\psi \neq 0$.

Exercise 4.6. Consider the space $\Lambda^2 V \subset V \otimes V$, identified with the Lie algebra $\mathfrak{so}(V)$, and let $\Lambda^2 V \subset V \otimes V \xrightarrow{\sigma} \mathrm{Cl}(V)$ be the Clifford multiplication map. Prove that it gives a Lie algebra embedding $\mathfrak{so}(V) \longrightarrow \mathrm{Cl}(V)$.