# Class assignment 5: Hypercomplex manifolds

**Exercise 5.1.** Let $(M, I, J, K)$ be a hypercomplex manifold, $d_I := IdI^{-1}, d_J := JdJ^{-1}, d_K := KdK^{-1}$, and $D := dd_Id_Jd_K : \Lambda^*(M) \longrightarrow \Lambda^{*+4}(M)$. Prove that $D$ is independent from the choice of a basis $I, J, K$ in quaternions.

**Exercise 5.2.** Let $(M, I, J, K)$ be a hypercomplex manifold, and $\nabla$ a torsion-free connection preserving $I, J, K$ (such connection is called the Obata connection). Consider $M$ as a complex manifold $(M, I)$. Prove that $\nabla^{0,1} = \bar\partial$ on $\Lambda_I^{p,0}(M)$, where

$$\bar\partial : \Lambda_I^{p,0}(M) \longrightarrow \Lambda_I^{p,1}(M) = \Lambda_I^{p,0}(M) \oplus \Lambda_I^{0,1}(M)$$

Prove that $\nabla_X^{1,0}(\eta) = J(\partial J^{-1}(\eta) \lrcorner X)$ for any $\eta \in \Lambda_I^{p,0}(M)$ and $X \in T_I^{1,0}(M)$.

**Exercise 5.3.** Construct a nilpotent Lie group with a non-trivial left invariant hypercomplex structure.

**Definition 5.1.** Let $(M, I, J, K)$ be a hypercomplex manifold, and $g$ a quaternionic Hermitian metric. Consider the corresponding 2-form $\Omega := \omega_J + \sqrt{-1}\omega_K \in \Lambda^{2,0}(M, I)$. The metric $g$ is called HKT if $\partial\Omega = 0$, where $\partial : \Lambda^{2,0}(M, I) \longrightarrow \Lambda^{3,0}(M, I)$ is the standard Hodge differential.

**Exercise 5.4.** Let $(M, I, J, K)$ be a hypercomplex manifold, $g$ a quaternionic Hermitian metric, and $\Omega := \omega_J + \sqrt{-1}\omega_K \in \Lambda^{2,0}(M, I)$. Prove that $\partial\Omega = 0$ if and only if $d\omega_I$ has weight 1 with respect to the natural $SU(2)$-action on 3-forms.

**Exercise 5.5.** Let $g$ be a Kähler metric on $(M, I)$, where $(M, I, J, K)$ is hypercomplex, and $g_1$ be $g$ averaged with the natural $SU(2)$-action. Prove that $g_1$ is HKT.