Class assignment 6: Stable bundles and Yang-Mills connections

Exercise 6.1. Find an example of compact Kähler manifold M such that its tangent bundle is

a. unstable

b. unstable for any Kähler structure on M.

Exercise 6.2. Let F and G be coherent sheaves on a compact Kähler manifold M. Assume that F and G admit filtrations $0 = F_0 \subset F_1 \subset ... \subset F_n = F$ $0 = G_0 \subset G_1 \subset ... \subset G_n = G$ with semistable subquotients, and $\mathsf{slope}(F_i/F_{i-1}) < \mathsf{slope}(G_j/G_{j-1})$ for all i, j. Prove that $\operatorname{Hom}(F, G) = 0$.

Exercise 6.3. Let M be a compact Kähler manifold with unstable tangent bundle. Suppose that $\mathsf{slope}(TM) \geqslant 0$. Prove that there exists a foliation $\mathcal{F}$ on M such that its tangent sheaf is destabilizing.

Exercise 6.4. Let (B, ∇) be a holomorphic Hermitian vector bundle on compact Kähler manifold with Chern connection and curvature $\Theta \in \Lambda^{1,1}(M) \otimes \operatorname{End} B$ such that $\Lambda\Theta = f \operatorname{\mathsf{Id}}_B$, where f is a function. Prove that $f = \mathsf{const}$.

Exercise 6.5. Let L be a holomorphic line bundle on a compact Hermitian manifold (M, I, ω) (not necessarily Kähler). Prove that there exists a constant $c \in \sqrt{-1}\mathbb{R}$ and a metric h on L such that the curvature of the Chern connection on L satisfies $\Lambda_\omega(\Theta) = c$. Prove that h is unique up to a constant.