Hyperkahler manifolds,

lecture 2: Calabi-Yau theorem

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Holomorphic vector bundles

DEFINITION: A $\overline{\partial}$-operator on a smooth bundle is a map $V \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M)\otimes V$, satisfying $\overline{\partial}(fb) = \overline{\partial}(f)\otimes b + f\overline{\partial}(b)$ for all $f \in C^\infty M, b \in V$.

REMARK: A $\overline{\partial}$-operator on B can be extended to

$$\overline{\partial}: \Lambda^{0,i}(M)\otimes V \longrightarrow \Lambda^{0,i+1}(M)\otimes V,$$

using $\overline{\partial}(\eta\otimes b) = \overline{\partial}(\eta)\otimes b + (-1)^{\tilde{\eta}}\eta\wedge\overline{\partial}(b)$, where $b \in V$ and $\eta \in \Lambda^{0,i}(M)$.

DEFINITION: A holomorphic vector bundle on a complex manifold (M, I) is a vector bundle equipped with a $\overline{\partial}$-operator which satisfies $\overline{\partial}^2 = 0$. In this case, $\overline{\partial}$ is called **a holomorphic structure operator**.

EXERCISE: Consider the Dolbeault differential $\overline{\partial}: \Lambda^{p,0}(M) \longrightarrow \Lambda^{p,1}(M) = \Lambda^{p,0}(M)\otimes\Lambda^{0,1}(M)$. **Prove that it is a holomorphic structure operator on $\Lambda^{p,0}(M)$.**

DEFINITION: The corresponding holomorphic vector bundle $(\Lambda^{p,0}(M), \overline{\partial})$ is called **the bundle of holomorphic p-forms**, denoted by $\Omega^p(M)$.

REMINDER: Chern connection

DEFINITION: Let (B, ∇) be a smooth bundle with connection and a holomorphic structure $\overline{\partial}\ B \longrightarrow \Lambda^{0,1}(M) \otimes B$. Consider a Hodge decomposition of ∇, $\nabla = \nabla^{0,1} + \nabla^{1,0}$,

$$\nabla^{0,1} : V \longrightarrow \Lambda^{0,1}(M) \otimes V, \quad \nabla^{1,0} : V \longrightarrow \Lambda^{1,0}(M) \otimes V.$$

We say that ∇ is **compatible with the holomorphic structure** if $\nabla^{0,1} = \overline{\partial}$.

DEFINITION: An Hermitian holomorphic vector bundle is a smooth complex vector bundle equipped with a Hermitian metric and a holomorphic structure operator $\overline{\partial}$.

DEFINITION: A Chern connection on a holomorphic Hermitian vector bundle is a connection compatible with the holomorphic structure and preserving the metric.

THEOREM: On any holomorphic Hermitian vector bundle, **the Chern connection exists, and is unique.**

REMINDER: Curvature of a connection

DEFINITION: Let $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ be a connection on a smooth budnle. Extend it to an operator on B-valued forms

$$B \xrightarrow{\nabla} \Lambda^1(M) \otimes B \xrightarrow{\nabla} \Lambda^2(M) \otimes B \xrightarrow{\nabla} \Lambda^3(M) \otimes B \xrightarrow{\nabla} ...$$

using $\nabla(\eta \otimes b) = d\eta + (-1)^{\tilde{\eta}}\eta \wedge \nabla b$. The operator $\nabla^2 : B \longrightarrow B \otimes \Lambda^2(M)$ is called **the curvature** of ∇. The operator $\nabla : \Lambda^i(M) \otimes B \xrightarrow{\nabla} \Lambda^{i+1}(M) \otimes B$ **is often denoted d_∇.**

REMARK: The algebra of $\mathrm{End}(B)$-valued forms naturally acts on $\Lambda^* M \otimes B$. The curvature satisfies $\nabla^2(fb) = d^2 fb + df \wedge \nabla b - df \wedge \nabla b + f\nabla^2 b = f\nabla^2 b$, hence it is $C^\infty M$-linear. **We consider it as an $\mathrm{End}(B)$-valued 2-form on M.**

REMARK: (Bianchi identity)
Clearly, $[\nabla, \nabla^2] = [\nabla^2, \nabla] + [\nabla, \nabla^2] = 0$, hence $[\nabla, \nabla^2] = 0$. This gives **the Bianchi identity:** $d_\nabla(\Theta_B) = 0$, where Θ is considered as a section of $\Lambda^2(M) \otimes \mathrm{End}(B)$, and $d_\nabla : \Lambda^2(M) \otimes \mathrm{End}(B) \longrightarrow \Lambda^3(M) \otimes \mathrm{End}(B)$ the operator defined above.

REMINDER: Curvature of a holomorphic line bundle

REMARK: If B is a line bundle, $\operatorname{End} B$ is trivial, and **the curvature Θ_B of B is a closed 2-form.**

DEFINITION: Let ∇ be a unitary connection in a line bundle. The cohomology class $c_1(B) := \frac{\sqrt{-1}}{2\pi}[\Theta_B] \in H^2(M)$ is called **the real first Chern class of a line bunlde B**.

An exercise: Check that $c_1(B)$ is independent from a choice of ∇.

REMARK: When speaking of a **"curvature of a holomorphic bundle"**, one usually means the curvature of a Chern connection.

REMARK: Let B be a holomorphic Hermitian line bundle, and b its non-degenerate holomorphic section. Denote by η a (1,0)-form which satisfies $\nabla^{1,0}b = \eta \otimes b$. Then $d|b|^2 = \operatorname{Re} g(\nabla^{1,0}b, b) = \operatorname{Re} \eta |b|^2$. **This gives $\nabla^{1,0}b = \frac{\partial |b|^2}{|b|^2} b = 2\partial \log |b| b$.**

REMARK: Then $\Theta_B(b) = 2\bar\partial\partial \log |b| b$, **that is, $\Theta_B = -2\partial\bar\partial \log |b|$.**

COROLLARY: If $g' = e^{2f}g$ – two metrics on a holomorphic line bundle, Θ, Θ' their curvatures, **one has $\Theta' - \Theta = -2\partial\bar\partial f$**

$\partial\bar\partial$-lemma

THEOREM: ("$\partial\bar\partial$-lemma")
Let M be a compact Kaehler manifold, and $\eta \in \Lambda^{p,q}(M)$ an exact form. Then $\eta = \partial\bar\partial\alpha$, for some $\alpha \in \Lambda^{p-1,q-1}(M)$.

Its proof uses Hodge theory.

COROLLARY: Let (L, h) be a holomorphic line bundle on a compact complex manifold, Θ its curvature, and η a (1,1)-form in the same cohomology class as $[\Theta]$. **Then there exists a Hermitian metric h' on L such that its curvature is equal to η.**

Proof: Let Θ' be the curvature of the Chern connection associated with h'. Then $\Theta' - \Theta = -2\partial\bar\partial f$, wgere $f = \log(h'h^{-1})$. Then $\Theta' - \Theta = \eta - \Theta = -2\partial\bar\partial f$ **has a solution f by $\partial\bar\partial$-lemma,** because $\eta - \Theta$ is exact. ■

Calabi-Yau manifolds

REMARK: Let B be a line bundle on a manifold. Using the long exact sequence of cohomology associated with the exponential sequence

$$0 \longrightarrow \mathbb{Z}_M \longrightarrow C^\infty M \longrightarrow (C^\infty M)^* \longrightarrow 0,$$

we obtain $0 \longrightarrow H^1(M, (C^\infty M)^*) \longrightarrow H^2(M, \mathbb{Z}) \longrightarrow 0$**.**

DEFINITION: Let B be a complex line bundle, and ξ_B its defining element in $H^1(M, (C^\infty M)^*)$. Its image in $H^2(M, \mathbb{Z})$ is called **the integer first Chern class** of B, denoted by $c_1(B, \mathbb{Z})$ or $c_1(B)$.

REMARK: A complex line bundle B is (topologically) trivial if and only if $c_1(B, \mathbb{Z}) = 0$.

THEOREM: (Gauss-Bonnet) A real Chern class of a vector bundle **is an image of the integer Chern class $c_1(B, \mathbb{Z})$** under the natural homomorphism $H^2(M, \mathbb{Z}) \longrightarrow H^2(M, \mathbb{R})$.

DEFINITION: A first Chern class of a complex n-manifold is $c_1(\Lambda^{n,0}(M))$.

DEFINITION:
A Calabi-Yau manifold is a compact Kaehler manifold with $c_1(M, \mathbb{Z}) = 0$.

Ricci form of a Kähler manifold

THEOREM: (Bogomolov) Let M be a compact Kähler n-manifold with $c_1(M,\mathbb{Z}) = 0$. **Then the canonical bundle $K_M := \Omega^n(M)$ is trivial.**

Proof: Follows from the Calabi-Yau theorem (later today). ■

In other words, a manifold is Calabi-Yau if and only if its canonical bundle is trivial.

DEFINITION: Let (M,ω) be a Kähler manifold. The metric on K_M can be written as $|\Omega|^2 = \frac{\Omega\wedge\overline{\Omega}}{\omega^n}$. The **Ricci form** on M is the curvature of the Chern connection on K_M. The manifold M is **Ricci-flat** if its Ricci form vanishes.

REMARK: Since a canonical bundle K_M of a Calabi-Yau manifold is trivial, it admits a metric with trivial connection. Calabi conjectured that **this metric on K_M is induced by a Kähler metric ω on M** and proved that such a metric is unique for any cohomology class $[\omega] \in H^{1,1}(M,\mathbb{R})$. Yau proved that it always exists.

DEFINITION: A Ricci-flat Kähler metric is called **Calabi-Yau metric**.

Calabi-Yau theorem and Monge-Ampère equation

REMARK: Let (M,ω) be a Kähler n-fold, and Ω a non-degenerate section of $K(M)$, Then $|\Omega|^2 = \frac{\Omega\wedge\overline{\Omega}}{\omega^n}$. If ω_1 is a new Kaehler metric on (M,I), h, h_1 the associated metrics on $K(M)$, then $\frac{h}{h_1} = \frac{\omega_1^n}{\omega^n}$.

REMARK: For two metrics ω_1, ω in the same Kähler class, one has $\omega_1 - \omega = dd^c\varphi$, **for some function** φ (dd^c-lemma).

COROLLARY: Let M be a Calabi-Yau manifold, ω its Kähler form, Ω a non-degenerate section of the canonical bundle. A metric $\omega_1 = \omega + \partial\overline{\partial}\varphi$ **is Ricci-flat if and only if** $(\omega + dd^c\varphi)^n = \omega^n e^f$, where $-2\partial\overline{\partial} f = \Theta_{K,\omega}$ **(such** f **exists by** $\partial\overline{\partial}$**-lemma).**

Proof. Step 1: For f such that $-2\partial\overline{\partial} f = \Theta_{K,\omega}$, the curvature of the metric $h \longrightarrow \frac{h\wedge\overline{h}}{\omega^n e^f}$ on K_M is equal to $\Theta_{K,\omega} + 2\partial\overline{\partial} f = 0$.

Proof. Step 2: ω_1 **is Ricci-flat if and only if the induced metric on** K_M **is flat,** which is equivalent to $(\omega + dd^c\varphi)^n = \omega^n e^f$. ■

To find a Ricci-flat metric **it remains to solve an equation** $(\omega + dd^c\varphi)^n = \omega^n e^f$ **for a given** f**.**

The complex Monge-Ampère equation

To find a Ricci-flat metric **it remains to solve an equation** $(\omega + dd^c\varphi)^n = \omega^n e^f$ **for a given** f**.**

THEOREM: (Calabi-Yau) Let (M,ω) be a compact Kaehler n-manifold, and f any smooth function. **Then there exists a unique up to a constant function** φ such that $(\omega + \sqrt{-1}\,\partial\bar\partial\varphi)^n = Ae^f\omega^n$, where A is a positive constant obtained from the formula $\int_M Ae^f\omega^n = \int_M \omega^n$.

DEFINITION:

$$(\omega + \sqrt{-1}\,\partial\bar\partial\varphi)^n = Ae^f\omega^n,$$

is called **the Monge-Ampere equation.**

Uniqueness of solutions of complex Monge-Ampere equation

PROPOSITION: (Calabi) **A complex Monge-Ampere equation has at most one solution,** up to a constant.

Proof. Step 1: Let ω_1, ω_2 be solutions of Monge-Ampere equation. Then $\omega_1^n = \omega_2^n$. By construction, one has $\omega_2 = \omega_1 + \sqrt{-1}\,\partial\bar\partial\psi$. **We need to show** $\psi = const$**.**

Step 2: $\omega_2 = \omega_1 + \sqrt{-1}\,\partial\bar\partial\psi$ gives

$$0 = (\omega_1 + \sqrt{-1}\,\partial\bar\partial\psi)^n - \omega_1^n = \sqrt{-1}\,\partial\bar\partial\psi \wedge \sum_{i=0}^{n-1} \omega_1^i \wedge \omega_2^{n-1-i}.$$

Step 3: Let $P := \sum_{i=0}^{n-1} \omega_1^i \wedge \omega_2^{n-1-i}$. This is a strictly positive $(n-1, n-1)$-form. **There exists a Hermitian form ω_3 on M such that $\omega_3^{n-1} = P$.**

Step 4: Since $\sqrt{-1}\,\partial\bar\partial\psi \wedge P = 0$, this gives $\psi\partial\bar\partial\psi \wedge P = 0$. Stokes' formula implies

$$0 = \int_M \psi \wedge \partial\bar\partial\psi \wedge P = -\int_M \partial\psi \wedge \bar\partial\psi \wedge P = -\int_M |\partial\psi|_3^2 \omega_3^n.$$

where $|\cdot|_3$ is the metric associated to ω_3. **Therefore $\bar\partial\psi = 0$.** ∎

Bochner's vanishing

THEOREM: (Bochner vanishing theorem) On a compact Ricci-flat Calabi-Yau manifold, **any holomorphic p-form η is parallel** with respect to the Levi-Civita connection: $\nabla(\eta) = 0$.

REMARK: Its proof is based on spinors: η gives a harmonic spinor, and **on a Ricci-flat Riemannian spin manifold, any harmonic spinor is parallel.**

DEFINITION: A **holomorphic symplectic manifold** is a manifold admitting a non-degenerate, holomorphic symplectic form.

REMARK: A holomorphic symplectic manifold is Calabi-Yau. The top exterior power of a holomorphic symplectic form **is a non-degenerate section of canonical bundle.**

Hyperkähler manifold

REMARK: Due to Bochner's vanishing, **holonomy of Ricci-flat Calabi-Yau manifold lies in $SU(n)$**, and **holonomy of Ricci-flat holomorphically symplectic manifold lies in $Sp(n)$** (a group of complex unitary matrices preserving a complex-linear symplectic form).

DEFINITION: A holomorphically symplectic Kähler manifold with holonomy in $Sp(n)$ is called **hyperkähler**.

REMARK: Since $Sp(n) = SU(\mathbb{H}, n)$, a **hyperkähler manifold admits quaternionic action in its tangent bundle.**

EXAMPLES.

EXAMPLE: An even-dimensional complex vector space.

EXAMPLE: An even-dimensional complex torus.

EXAMPLE: A non-compact example: $T^*\mathbb{C}P^n$ (Calabi).

REMARK: $T^*\mathbb{C}P^1$ **is a resolution of a singularity** $\mathbb{C}^2/\pm 1$**.**

REMARK: Let M be a 2-dimensional complex manifold with holomorphic symplectic form outside of singularities, which are all of form $\mathbb{C}^2/\pm 1$. Then **its resolution is also holomorphically symplectic.**

EXAMPLE: Take a 2-dimensional complex torus T, then all the singularities of $T/\pm 1$ are of this form. Its resolution $\widetilde{T/\pm 1}$ is called **a Kummer surface**. **It is holomorphically symplectic**.

REMARK: Take a symmetric square $\operatorname{Sym}^2 T$, with a natural action of T, and let $T^{[2]}$ be a blow-up of a singular divisor. **Then $T^{[2]}$ is naturally isomorphic to the Kummer surface $\widetilde{T/\pm 1}$.**

K3 surfaces

DEFINITION: A K3-surface is a deformation of a Kummer surface.

"K3: Kummer, Kähler, Kodaira" (a name is due to A. Weil).

"Faichan Kangri (K3) is the 12th highest mountain on Earth."

THEOREM: Any complex compact surface with $c_1(M) = 1$ and $H^1(M) = 0$ **is isomorphic to K3.** Moreover, **it is hyperkähler.**

Hilbert schemes

REMARK: A complex surface is a 2-dimensional complex manifold.

DEFINITION: A **Hilbert scheme** $M^{[n]}$ of a complex surface M is a classifying space of all ideal sheaves $I \subset \mathcal{O}_M$ for which the quotient $\mathcal{O}_M/I$ has dimension n over $\mathbb{C}$.

REMARK: A Hilbert scheme **is obtained as a resolution of singularities** of the symmetric power $\operatorname{Sym}^n M$.

THEOREM: (Fujiki, Beauville) **A Hilbert scheme of a hyperkähler surface is hyperkähler.**

EXAMPLE: A Hilbert scheme of K3.

EXAMPLE: Let T is a torus. Then it acts on its Hilbert scheme freely and properly by translations. For $n = 2$, the quotient $T^{[n]}/T$ is a Kummer K3-surface. For $n > 2$, it is called **a generalized Kummer variety**.

REMARK: There are 2 more "sporadic" examples of compact hyperkähler manifolds, constructed by K. O'Grady. **All known compact hyperkaehler manifolds are these 2 and the three series:** tori, Hilbert schemes of K3, and generalized Kummer.