Hyperkahler manifolds,

lecture 3: Spinors

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Clifford algebras

DEFINITION: The Clifford algebra of a vector space V with a scalar product q is an algebra generated by V with a relation $xy+yx = q(x,y)1$, that is, a quotient of $T^{\otimes}V := k \oplus V \oplus V \otimes V \oplus ... \oplus T^{\otimes i}V$ by an ideal generated by $xy + yx = -2g(x,y)$ for all $x, y \in V$.

EXAMPLE: If $g = 0$, Clifford algebra is Grassmann algebra.

CLAIM: $\dim \mathrm{Cl}(V,g) = 2^{\dim V}$.

Proof: Consider $\mathrm{Cl}(V,g)$ as a filtered algebra with r-th term of filtration given by r-th power of $V \subset \mathrm{Cl}(V)$. **Its associated graded algebra is Grassmann algebra.** ■

REMARK: The Clifford algebra is $\mathbb{Z}/2\mathbb{Z}$ graded: $\mathrm{Cl}(V,g) = \mathrm{Cl}_{\text{even}}(V,g) \oplus \mathrm{Cl}_{\text{odd}}(V,g)$.

DEFINITION: Let $A = A_{\text{even}} \oplus A_{\text{odd}}$ be a graded associative algebra. Let $A^{\perp}$ be the same vector space with new multiplication $a \bullet a' := (-1)^{\tilde{a}\tilde{a}'} aa'$.

EXERCISE: Prove that $\mathrm{Cl}(V,g)^{\perp} = \mathrm{Cl}(V,-g)$.

Graded tensor product

DEFINITION: Let $A := A_{\text{even}} \oplus A_{\text{odd}}$, $B := B_{\text{even}} \oplus B_{\text{odd}}$ be graded associative algebras. Define **the graded tensor product** $A\tilde{\otimes}B$ as $A \otimes B$ with multiplication given by $a \otimes b \cdot a' \otimes b' = (-1)^{\tilde{b}\tilde{a}'} aa' \otimes bb'$, where $\tilde{x}$ denotes the parity of x.

EXAMPLE: Graded tensor product of Grassmann algebras **gives the Grassmann algebra of a direct sum:**

$$\Lambda^* V \tilde{\otimes} \Lambda^* W \cong \Lambda^*(V \oplus W)$$

EXAMPLE: The same is true for Clifford algebras:

$$\mathrm{Cl}(V, g) \tilde{\otimes} \mathrm{Cl}(V', g') = \mathrm{Cl}(V \oplus V', g + g').$$

Pseudoscalar

LEMMA (*): Let $A := A_{\text{even}} \oplus A_{\text{odd}}$, $B := B_{\text{even}} \oplus B_{\text{odd}}$ be graded associative algebras. Suppose that B contains an even element **(pseudoscalar)** ε with the following properties:

$$\varepsilon^2 = 1, \varepsilon b = (-1)^{\tilde{b}} b\varepsilon.$$

Then $A \tilde{\otimes} B \cong A \otimes B$ (the graded tensor product is isomorphic to the usual one).

Proof: Consider a subalgebra $A' \subset A \tilde{\otimes} B$ generated by elements $a \tilde{\otimes} \varepsilon^{\tilde{a}}$ and $B' = 1 \otimes B \subset A \tilde{\otimes} B$. Then

1. $A' \cong A$ **commutes with** $B' \cong B$**.**

2. $A' \otimes B' = A \tilde{\otimes} B$ **as a vector space.** ■

REMARK (*): If in the definition of pseudoscalar we replace $\varepsilon^2 = 1$ by $\varepsilon^2 = -1$, **Lemma (*) will give** $A \tilde{\otimes} B \cong A^{\perp} \otimes B$**.**

Unit pseudoscalar

Let (V, g) be an oriented real vector space with orthogonal basis $e_1, ..., e_n$ such that $g(e_i, e_i) = \pm 1$. **Unit pseudoscalar** in $\mathrm{Cl}(V, g)$ is $\varepsilon := e_1 e_2 e_3 ... e_n$.

EXERCISE: **Prove $\varepsilon e_i = (-1)^{n-1} e_i \varepsilon$.**

EXERCISE: Prove that $\varepsilon^2 = (-1)^{\frac{(n-1)(n-2)}{2}}(-1)^q$ **if g has signature (p, q).**

REMARK: This gives

$$\varepsilon^2 = (-1)^{n(n-1)/2}(-1)^q = (-1)^{(p-q)(p-q-1)/2} = \begin{cases} +1 & p - q \equiv 0, 1 \mod 4 \\ -1 & p - q \equiv 2, 3 \mod 4. \end{cases}$$

DEFINITION: Denote the Clifford algebra of a real vector space of signature (p, q) by $\mathrm{Cl}(p, q)$.

COROLLARY: $\mathrm{Cl}(p+m, q+m') \cong \mathrm{Cl}(p, q) \otimes \mathrm{Cl}(m, m')$ **when $m+m'$ is even, and $m - m' \equiv 0 \mod 4$.**

Proof: The pseudoscalar ε in $\mathrm{Cl}(m, m')$ satisfies $\varepsilon^2 = 1$. Applying Lemma (*), we obtain $\mathrm{Cl}(p, q) \otimes \mathrm{Cl}(m, m') \cong \mathrm{Cl}(p, q) \tilde{\otimes} \mathrm{Cl}(m, m')$. Then we apply the isomorphism $\mathrm{Cl}(V, g) \tilde{\otimes} \mathrm{Cl}(V', g') = \mathrm{Cl}(V \oplus V', g + g')$. ■

Bott periodicity over $\mathbb{C}$

COROLLARY: Let $A[i]$ denote the tensor product $A \otimes \operatorname{Mat}(i, \mathbb{R}) \cong \operatorname{Mat}(i, A)$. **Then** $\operatorname{Cl}(p+1, q+1) \cong \operatorname{Cl}(p,q)[2]$**.**

Proof: Use the previous corollary and an isomorphism $\operatorname{Cl}(1,1) = \operatorname{Mat}(2, \mathbb{R})$ **(prove it).** ■

THEOREM: (Bott periodicity over $\mathbb{C}$)
Clifford algebra $\operatorname{Cl}(V, q)$ of a complex vector space $V = \mathbb{C}^n$ with q non-degenerate **is isomorphic to** $\operatorname{Mat}\left(\mathbb{C}^{n/2}\right)$ **(n even) and** $\operatorname{Mat}\left(\mathbb{C}^{\frac{n-1}{2}}\right) \oplus \operatorname{Mat}\left(\mathbb{C}^{\frac{n-1}{2}}\right)$ **(n odd).**

Proof: Use the previous corollary and isomorphisms $\operatorname{Cl}(\mathbb{C}) = \mathbb{C} \oplus \mathbb{C}$, $\operatorname{Cl}(0) = \mathbb{C}$.
■

Bott periodicity over $\mathbb{R}$.

COROLLARY: $\mathsf{Cl}(p+m,q+m') \cong \mathsf{Cl}(q,p) \otimes \mathsf{Cl}(m,m')$ **if** $m+m'$ **is even, and** $m-m' \equiv 2 \mod 4$.

Proof: In $\mathsf{Cl}(m,m')$ the pseudoscalar ε satisgies $\varepsilon^2 = -1$. Applying Remark (*), we obtain $\mathsf{Cl}(p,q)^\perp \otimes \mathsf{Cl}(m,m') \cong \mathsf{Cl}(p,q)\tilde{\otimes}\mathsf{Cl}(m,m') \cong \mathsf{Cl}(p+m,q+m')$. Then we use an isomorphism $\mathsf{Cl}(p,q)^\perp = \mathsf{Cl}(p,q)$. ■

COROLLARY: $\mathsf{Cl}(p+2,q) \cong \mathsf{Cl}(q,p)[2]$ **and** $\mathsf{Cl}(p,q+2) \cong \mathsf{Cl}(q,p) \otimes \mathbb{H}$.

Proof: We use the previous corollary and the isomorphisms $\mathsf{Cl}(2,0) = \mathrm{Mat}(2,\mathbb{R})$, $\mathsf{Cl}(0,2) = \mathbb{H}$. ■

COROLLARY: (Bott Periodicity modulo 4):
The previous corollary immediately gives $\mathsf{Cl}(p+4,q) \cong \mathsf{Cl}(q,p+2)[2] = \mathsf{Cl}(p,q) \otimes \mathrm{Mat}(2,\mathbb{H})$ and $\mathsf{Cl}(p,q+4) \cong \mathsf{Cl}(q+2,p) \otimes \mathbb{H} = \mathsf{Cl}(p,q) \otimes \mathrm{Mat}(2,\mathbb{H})$.

Bott periodicity over $\mathbb{R}$ (2).

EXERCISE: Prove the isomorphism $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} = \mathrm{Mat}(4, \mathbb{R})$.

COROLLARY: (Bott Periodicity modulo 8):
This isomorphism and the previous corollary give $\mathrm{Cl}(p+8, q) = \mathrm{Cl}(p, q)[16]$, $\mathrm{Cl}(p, q+8) = \mathrm{Cl}(p, q)[16]$.

	1	2	3	4	5	6	7	8
$\mathrm{Cl}(i,0)$	$\mathbb{R}^2$	$\mathbb{R}[2]$	$\mathbb{C}[2]$	$\mathbb{H}[2]$	$\mathbb{H}[2] \oplus \mathbb{H}[2]$	$\mathbb{H}[4]$	$\mathbb{C}[8]$	$\mathbb{R}[16]$
$\mathrm{Cl}(0,i)$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}[2]$	$\mathbb{C}[4]$	$\mathbb{R}[8]$	$\mathbb{R}[8] \oplus \mathbb{R}[8]$	$\mathbb{R}[16]$

Pseudoscalar on an odd-dimensional space

DEFINITION: For any odd-dimensional space V, the pseudoscalar $\varepsilon = e_1 e_2 ... e_{2n+1}$ commutes with a multiplication by generators of $\mathrm{Cl}(V)$, hence defines an automorphism of $\mathrm{Cl}(V)$. If V were a complex vector space, we can always chose the basis $e_1, e_2, ..., e_{2n+1}$ in such a way that $\varepsilon^2 = 1$. **This gives the eigenvalue decomposition $\mathrm{Cl}(V) = \mathrm{Cl}^+(V) \oplus \mathrm{Cl}^-(V)$.**

CLAIM: Each of the algebras $\mathrm{Cl}^+(V)$, $\mathrm{Cl}^-(V)$ is isomorphic to $\mathrm{Mat}(\mathbb{C}^r)$.

Proof: Eigenvalues of ε acting on $\mathrm{Cl}(V)$ are equal to ± 1 because $\varepsilon^2 = 1$. On the other hand, an automorphism of V which exchanges e_1 and e_2 maps ε to $-\varepsilon$, hence permutes the eigenspaces. Therefore, **the subalgebras $\mathrm{Cl}^+(V)$, $\mathrm{Cl}^-(V)$ are isomorphic.** We obtain that **the decomposition $\mathrm{Cl}(V) = \mathrm{Mat}(2^n, \mathbb{C}) \oplus \mathrm{Mat}(2^n, \mathbb{C})$ coincides with the eigenspace decomposition defined by ε.**

REMARK: The center of $\mathrm{Cl}(V)$ is isomorphic to $\mathbb{C} \oplus \mathbb{C}$. **The orthogonal group $O(V)$ acts on $\mathrm{Cl}(V)$ and on $\mathbb{Z}$ by automorphisms, and maps ε to $\pm\varepsilon$.** In particular, **$SO(V)$ acts on $\mathrm{Cl}^{\pm}(V)$ by automorphisms**.

Automorphisms of matrix algebra

EXERCISE: Let V be a vector space over a field of characteristic 0. Prove that **the automorphism group** $\operatorname{Aut}(\operatorname{Mat}(V))$ **is isomorphic to** $PGL(V)$ (the quotient of $GL(V)$ by its center).

Spinorial group Spin$(2n)$

DEFINITION: Let $V = \mathbb{C}^{2n}$ be a vector space over $\mathbb{C}$ with non-degenerate scalar product. The group $SO(V)$ acts on Cl(V) be automorphisms, giving an action

$$SO(V) \hookrightarrow \operatorname{Aut}(\operatorname{Mat}(2^n, \mathbb{C})) = PGL(2^n, \mathbb{C})$$

as shown above.

DEFINITION: (Elie Cartan, 1913)
Spinor representation of the Lie algebra $\mathfrak{so}(V)$ is its representation on $\mathbb{C}^{2^n}$ induced by the isomorphism $\mathfrak{pgl}(2^n) = \mathfrak{sl}(2^n)$.

DEFINITION: Spinor group Spin$(2n+1)$ is a double cover of $SO(2n+1)$ obtained as a Lie group of $\mathfrak{so}(V)$ acting on its spinorial representation.

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EXERCISE: In even- and odd-dimensional case, prove that Spin(r) **is, indeed, a double cover of $SO(r)$.**

Principal bundles

DEFINITION: Let G be a Lie group. **Principal G-bundle** over a manifold M is a smooth fibration $P \mapsto M$ with a smooth G-action which acts freely and transitively on fibers.

EXAMPLE: Frame bundle on a smooth n-manifold M is the bundle of all frames (basises) in T_xM, for all $x \in M$.

DEFINITION: Let $H \longrightarrow G$ be a group homomorphism, and P a principal H-bundle. Then the quotient $P_G := P \times G/H$ (with H acting on both components in a natural way) is called **an associated principal bundle**, and P is called **reduction of the principal G-bundle P_G to the group H**.

DEFINITION: Let G be a Lie group, and $G \longrightarrow GL(n, \mathbb{R})$ a group homomorphism. **A G-structure on a manifold M** is a reduction of the principal frame bundle to G.

DEFINITION: Let G be a Lie group, V its representation, and P a principal G-bundle on M. The quotient $P \times V/G$ is a vector bundle over M, called **the associated vector bundle**.

Spin-structures and spinor bundles

DEFINITION: A spin-structure on an oriented n-manifold M is a reduction of its structure group to Spin(n). A manifold is called **spin** if it admits a spin-structure.

REMARK: This happens precisely when the second Stiefel-Whitney class $w_2(M)$ vanishes.

DEFINITION: A bundle of spinors on a spin-manifold M is a vector bundle associated to the principal $Spin(n)$-bundle and a spin representation.

REMARK: The Levi-Civita connection **is naturally extended from a connection on the bundle of orthogonal frames to its double cover.** This defines the Levi-Civita connection on the spinor bundle.

Spinor bundles and Dirac operator

DEFINITION: Consider the map $TM \otimes \mathrm{Spin} \longrightarrow \mathrm{Spin}$ induced by the Clifford multiplication. One defines **the Dirac operator** $D : \mathrm{Spin} \longrightarrow \mathrm{Spin}$ as a composition of $\nabla : \mathrm{Spin} \longrightarrow \Lambda^1 M \otimes \mathrm{Spin} = TM \otimes \mathrm{Spin}$ and the multiplication.

DEFINITION: A **harmonic spinor** is a spinor ψ such that $D(\psi) = 0$.

THEOREM: (Bochner's vanishing) A harmonic spinor ψ on a compact manifold with vanishing scalar curvature $Sc = Tr(\mathrm{Ric})$ **satisfies $\nabla\psi = 0$.**

Proof: The **coarse Laplacian** $\nabla^*\nabla$ is expressed through the Dirac operator using the **Lichnerowitz formula** $\nabla^*\nabla - D^2 = -\frac{1}{4}Sc$. When these two operators are equal, **any harmonic spinor ψ lies in $\ker \nabla^*\nabla$, giving $(\psi, \nabla^*\nabla\psi) = (\nabla\psi, \nabla\psi) = 0$.** ■

Bochner's vanishing on Kaehler manifolds

REMARK: A Kaehler manifold is spin if and only if $c_1(M)$ is even, or, equivalently, if there exists a square root of a canonical bundle $K^{1/2}$.

REMARK: On a Kaehler manifold of complex dimension n, **one has a natural isomorphism between the spinor bundle and $\Lambda^{*,0}(M) \otimes K^{1/2}$** (for n even) and $\Lambda^{2*,0}(M) \otimes K^{1/2}$ (for n odd).

REMARK: On a Kähler manifold, the Dirac operator corresponds to $\partial + \partial^*$.

COROLLARY: On a Ricci-flat Kähler manifold, all $\alpha \in \ker(\partial+\partial^*)\Big|_{\Lambda^{*,0}(M)}$ ara parallel.

REMARK: $\ker \partial + \partial^* = \ker\{\partial, \partial^*\}$, where $\{\cdot, \cdot\}$ denotes the anticommutator. However, $\{\partial, \partial^*\} = \{\overline{\partial}, \overline{\partial}^*\}$ as Kähler identities imply. Therefore, **on a Kähler manifold, harmonic spinors are holomorphic forms**.

THEOREM: (Bochner's vanishing) Let M be a Ricci-flat Kaehler manifold, and $\Omega \in \Lambda^{p,0}(M)$ a holomorphic differential form. **Then $\nabla\Omega = 0$.**