Hyperkahler manifolds,

lecture 4: Bochner vanishing

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http://bogomolov-lab.ru/KURSY/HK-2019/

Clifford algebras (reminder)

DEFINITION: The Clifford algebra of a vector space V with a scalar product q is an algebra generated by V with a relation $xy + yx = -2q(x, y)1$, that is, a quotient of $T^{\otimes}V := k \oplus V \oplus V \otimes V \oplus ... \oplus T^{\otimes i}V$ by an ideal generated by $xy + yx = -2g(x, y)$ for all $x, y \in V$.

THEOREM: (Bott periodicity over $\mathbb{C}$)
Clifford algebra $\mathrm{Cl}(V, q)$ of a complex vector space $V = \mathbb{C}^n$ with q non-degenerate **is isomorphic to** $\mathrm{Mat}\left(\mathbb{C}^{n/2}\right)$ **(n even) and** $\mathrm{Mat}\left(\mathbb{C}^{\frac{n-1}{2}}\right) \oplus \mathrm{Mat}\left(\mathbb{C}^{\frac{n-1}{2}}\right)$ **(n odd).**

Spin(n,n): an explicit construction

Let $W = U \oplus V$ be a vector space with U, V dual and the quadratic form pairing (u,v) and (u',v') as follows $q((u,v),(u',v')) = \langle u, v'\rangle + \langle u', v\rangle$.

DEFINITION: Consider **the exterior multiplication** operator $e_u: \Lambda^*(U) \longrightarrow \Lambda^{*+1}(U)$ with $e_u(\alpha) = u \wedge \alpha$ and **the convolution operator** $i_v: \Lambda^*(U) \longrightarrow \Lambda^{*-1}(U)$, with $i_v(\alpha)(v_1, ..., v_k) = \alpha(v, v_1, ..., v_k)$.

CLAIM: These operators satisfy the following relations: $i_v, i_{v'}$ **anticommute for all** v, v'**;** $e_u, e_{u'}$ **anticommute for all** v, v'**; finally,** $\{i_v, e_u\} = \langle u, v\rangle \cdot \operatorname{Id}$**,** where $\{\cdot,\cdot\}$ (as usual) denotes the supercommutator, $\{a, b\} = ab - (-1)^{\tilde{a}\tilde{b}} ba$. ■

REMARK: These are the same relation as in Clifford algebra! **This defines a map** $\operatorname{Cl}(W) \longrightarrow \operatorname{Mat}(\Lambda^*(U))$**.**

EXERCISE: Fix a basis u_i in U, and let v_j be the dual basis in V. For any pair of monomials A, B in $\Lambda^*(U)$, **find a product of a sequence of of** i_{v_i}**,** e_{u_j} **which maps** A **to** B **and puts all other monomials to 0.**

CLAIM: The natural map $\operatorname{Cl}(W) \longrightarrow \operatorname{Mat}(\Lambda^*(U))$ **is an isomorphism.**

Proof: See the previous exercise. ■

Spinorial group Spin$(2n)$ (reminder)

EXERCISE: Let V be a vector space over a field of characteristic 0. Prove that **the automorphism group** Aut(Mat(V)) **is isomorphic to** $PGL(V)$ (the quotient of $GL(V)$ by its center).

DEFINITION: (Elie Cartan, 1913)
Spinor representation of the Lie algebra $\mathfrak{so}(2n)$ is its representation on $\mathbb{C}^{2^n}$ induced by the isomorphism $\mathfrak{pgl}(2^n) = \mathfrak{sl}(2^n) =$ Aut(Cl$(2n)$). **Spinor representation** of the Lie algebra $\mathfrak{so}(2n+1)$ is any of two representations of $\mathfrak{so}(2n+1)$ on $\mathbb{C}^{2^n}$ induced by the isomorphism $\mathfrak{pgl}(2^n) = \mathfrak{sl}(2^n) =$ Aut(Cl$^\pm(2n+1)$), where Cl$^{pm}(V) =$ Mat(2^n) is one of two components of Cl$(2n+1) =$ Mat$(\mathbb{C}^n) \oplus$ Mat$(\mathbb{C}^n)$,

DEFINITION: Spinor group Spin(k) is a double cover of $SO(k)$ obtained as a Lie group of $\mathfrak{so}(V)$ acting on its spinorial representation.

Principal bundles (reminder)

DEFINITION: Let G be a Lie group. **Principal G-bundle** over a manifold M is a smooth fibration $P \mapsto M$ with a smooth G-action which acts freely and transitively on fibers.

EXAMPLE: Frame bundle on a smooth n-manifold M is the bundle of all frames (basises) in T_xM, for all $x \in M$.

DEFINITION: Let $H \longrightarrow G$ be a group homomorphism, and P a principal H-bundle. Then the quotient $P_G := P \times G/H$ (with H acting on both components in a natural way) is called **an associated principal bundle**, and P is called **reduction of the principal G-bundle P_G to the group H**.

DEFINITION: Let G be a Lie group, and $G \longrightarrow GL(n, \mathbb{R})$ a group homomorphism. **A G-structure on a manifold M** is a reduction of the principal frame bundle to G.

DEFINITION: Let G be a Lie group, V its representation, and P a principal G-bundle on M. The quotient $P \times V/G$ is a vector bundle over M, called **the associated vector bundle**.

Spin-structures and spinor bundles (reminder)

DEFINITION: A spin-structure on an oriented n-manifold M is a reduction of its structure group to $\mathrm{Spin}(n)$. A manifold is called **spin** if it admits a spin-structure.

REMARK: This happens precisely when the second Stiefel-Whitney class $w_2(M)$ vanishes.

DEFINITION: A bundle of spinors on a spin-manifold M is a vector bundle associated to the principal $\mathrm{Spin}(n)$-bundle and a spin representation.

REMARK: The Levi-Civita connection **is naturally extended from a connection on the bundle of orthogonal frames to its double cover.** This defines the Levi-Civita connection on the spinor bundle.

Spin-structures on Calabi-Yau manifolds

REMARK: Let (M, I, g) be a Kähler manifold. The Hermitian form defines a pairing between the spaces $T^{1,0}(M)$ and $T^{0,1}(M)$, which are isotropic. **Therefore,** $\operatorname{Cl}(TM \otimes \mathbb{C}, g) = \operatorname{Mat}(\Lambda^{*,0}(M))$.

REMARK: The real structure on $\operatorname{Cl}(TM \otimes \mathbb{C})$ exchanges i_v and $e_{\overline{v}}$, hence it exchanges $\Lambda^{p,0}(M)$ and $\Lambda^{n-p,0}(M)$. Therefore, **the bundle $\Lambda^{*,0}(M)$ can be identified with spinors only for Calabi-Yau manifolds.**

CLAIM: For any Calabi-Yau manifold, **there is a spin structure such that $\Lambda^{*,0}(M)$ is a spinorial representation.**

Proof: To construct such a structure, we need to exhibit a real structure τ on $\Lambda^{*,0}(M)$ which is compatible with the real structure on $\operatorname{Cl}(\otimes\mathbb{C})$, that is, exchanging i_v and $e_{\overline{v}}$. For any $(p,0)$-form α, let $\tau(\alpha) := \overline{\frac{*\alpha}{\Theta}}$, where $\Theta \in \Lambda^{n,0}(M)$ is a parallel section which trivializes the canonical bundle. ■

Spinor bundles and Dirac operator (reminder)

DEFINITION: Consider the map $TM \otimes \text{Spin} \longrightarrow \text{Spin}$ induced by the Clifford multiplication. One defines **the Dirac operator** $D: \text{Spin} \longrightarrow \text{Spin}$ as a composition of $\nabla: \text{Spin} \longrightarrow \Lambda^1 M \otimes \text{Spin} = TM \otimes \text{Spin}$ and the multiplication.

DEFINITION: A **harmonic spinor** is a spinor ψ such that $D(\psi) = 0$.

THEOREM: (Bochner's vanishing)
A harmonic spinor ψ on a compact manifold with vanishing scalar curvature $Sc := Tr(\text{Ric})$ **satisfies $\nabla\psi = 0$.**

Proof: Later today.

Bochner's vanishing on Kähler manifolds

REMARK: **A Kähler manifold is spin if and only if $c_1(M)$ is even,** or, equivalently, if there exists a square root of a canonical bundle $K^{1/2}$.

REMARK: On a Kähler manifold of complex dimension n, **one has a natural isomorphism between the spinor bundle and $\Lambda^{*,0}(M)\otimes K^{1/2}$** (for n even) and $\Lambda^{2*,0}(M)\otimes K^{1/2}$ (for n odd).

REMARK: On a Kähler manifold, the Dirac operator corresponds to $\partial+\partial^*$.

COROLLARY: **On a Ricci-flat Kähler manifold, all $\alpha\in \ker(\partial+\partial^*)\Big|_{\Lambda^{*,0}(M)}$ ara parallel.**

REMARK: $\ker\partial+\partial^* = \ker\{\partial,\partial^*\}$, where $\{\cdot,\cdot\}$ denotes the anticommutator. However, $\{\partial,\partial^*\}=\{\bar\partial,\bar\partial^*\}$ as Kähler identities imply. Therefore, **on a Calabi-Yau manifold, harmonic spinors are holomorphic forms**.

THEOREM: **(Bochner's vanishing)** Let M be a Ricci-flat Kähler manifold, and $\Omega\in\Lambda^{p,0}(M)$ a holomorphic differential form. **Then $\nabla\Omega=0$.** ■

Gaussian curvature

CLAIM: Let ∇ be a Levi-Civita connection on a Riemannian manifold, and $R \in T^*M^{\otimes 3} \otimes TM$ its curvature tensor. Using an isomorphism $TM \cong T^*M$ given by the metric, we may consider R as an element in $T^*M^{\otimes 4}$. **Then R is a section of $\operatorname{Sym}^2(\Lambda^2 T^*M)$, antisymmetric in 1,2 and 3,4 indices.**

DEFINITION: Let V be a vectir space with non-degenerate scalar product g. **A trace** $\operatorname{Tr}_{12}: V^{\otimes^n} \longrightarrow V^{\otimes^{n-2}}$ is defined as a map dual to the multiplication $A \longrightarrow g \otimes A$. **The trace in i-th and j-th indices**, denoted as $\operatorname{Tr}_{ij}: V^{\otimes^n} \longrightarrow V^{\otimes^{n-2}}$, is defined as a map which acts in the i-th and j-th multiplier as Tr_{12} on the first two.

DEFINITION: Gaussian curvature of a Riemannian manifold is a scalar $\operatorname{Tr}_{13}\operatorname{Tr}_{24}(R)$, where R is the Riemannian curvature.

Clifford multiplication in $\mathsf{Sym}^2(\Lambda^2 V)$

LEMMA 1: Let $R \in \mathsf{Sym}^2(\Lambda^2 V)$, where V is a space with scalar product g. Denote the Clifford multiplication as $\sigma: V^{\otimes^4} \longrightarrow \mathsf{Cl}(V)$. **Then**

$$\sigma(R) = \mathsf{Tr}_{13}\mathsf{Tr}_{24} R + \sigma(\mathsf{Alt}(R)),$$

where $\mathsf{Alt}: \mathsf{Sym}^2(\Lambda^2 V) \longrightarrow \Lambda^4 V$ is the exterior product map.

Proof: Let $x, y, z, t \in V$, and $R(x, y, z, t) := (xy - yx)(zt - tz) + (zt - tz)(xy - yx)$ be the corresponding element in $\mathsf{Sym}^2(\Lambda^2 V)$. Then

1. If x, y, z, t are pairwise orthogonal, we have $\tau(R(x, y, z, t)) = \tau(\mathsf{Alt}(R))$, because x, y, z, t anticommute in the Clifford algebra.

2. If x, y, z are pairwise orthogonal, and $y = t$, then $xy - yx$ anticommutes with $zt - tz$, hence $\tau(R(x, y, z, t)) = 0$.

3. If x, y are orthogonal, $y = t$ and $x = z$, we have

$$\sigma(R(x, y, z, t)) = \sigma((xy - yx)^2) = g(x, x)g(y, y).$$

■

Laplacian and rough Laplacian

REMARK: Let $D: S \longrightarrow S$ be the Dirac operator, and $x_i \in TM$ an orthonormal frame. **Then $D(s) = \sum_i \sigma(x_i, \nabla_{x_i}(s))$, where $\sigma: TM \otimes S \longrightarrow S$ is Clifford multiplication.**

COROLLARY: Let $\Theta \in \Lambda^2 M \otimes \operatorname{End}(S)$ be the curvature of S. Then

$$D^2(s) = \sum_{i,j} \sigma(x_i x_j, \nabla_{x_i} \nabla_{x_j} s) = \sum_{i,j} \sigma(x_i x_j, \Theta_{x_i, x_j} s) + \sum_{i,j} \sigma(x_i x_j + x_j x_i, \nabla_{x_i} \nabla_{x_j} s).$$

Since $\sigma(x_i x_j + x_j x_i, v) = g(x_i, x_j) v$, this gives

$$D^2(s) = \sigma(\Theta, s) + \sum_i \nabla_{x_i} \nabla_{x_i} s.$$

DEFINITION: Rough Laplacian on a bundle B with connection on a Riemannian manifold is defined as $Đ(s) := \operatorname{Tr}_{12}(\nabla^2 s)$.

REMARK: The previous corollary is therefore rewritten as

$$D^2(s) = \sigma(\Theta, s) + Đ(s).$$

Weitzenböck formula

THEOREM: (Lichnerowicz-Weitzenböck formula)
Let M be a Riemannian manifold with spin structure, $Đ : S \longrightarrow S$ the rough Laplacian, Sc multiplication by the scalar product, and $D : S \longrightarrow S$ the Dirac operator. **Then $D^2 = Đ + \mathsf{Sc}$.**

Proof: $D^2(s) = \sigma(\Theta, s) + Đ(s)$, as shown above, and $\sigma(\Theta, s) = \mathsf{Sc}(s) + \sigma(\mathsf{Alt}(R))$ by Lemma 1. The last term vanishes, because $\mathsf{Alt}(R)$ (Bianchi identity). ■

REMARK: $g(Đ(s), s) = \mathsf{Tr}_{12}(\nabla^2(s), s) = g(\nabla(s), \nabla(s))$. This gives $\int g(Đ(s), s) = \int_M g(\nabla(s), \nabla(s))$. Therefore **on a compact manifold $Đ(s) = 0$ implies $\nabla(s) = 0$.**

Bochner vanishing for harmonic spinors

COROLLARY: (Bochner vanishing)
Let M be a compact Riemannian manifold with non-negative scalar curvature. **Then $\nabla(s) = 0$ for any harmonic spinor s.** If, in addition, $\mathrm{Sc} > 0$ somewhere on M, then $s = 0$.

Proof: Lichnerowicz-Weitzenböck formula gives

$$0 = g(D^2(s), s) = g(\text{Đ}(s), s) + \int_M \mathrm{Sc} \cdot g(s, s) = \int_M g(\nabla(s), \nabla(s)) + \int_M \mathrm{Sc} \cdot g(s, s).$$

The first term vanishes. Moreover, $s = 0$ on the set $U \subset M$ where $\mathrm{Sc} > 0$. Then $s = 0$ because $\nabla(s) = 0$. ■

Bochner's vanishing for holomorphic forms

THEOREM: On a compact Ricci-flat Calabi-Yau manifold, **any holomorphic p-form η is parallel** with respect to the Levi-Civita connection: $\nabla(\eta) = 0$.

Proof: Holomorphic forms are the same as harmonic spinors. ■

REMARK: The form η gives a harmonic spinor, and **on a Riemannian spin manifold with $\operatorname{Sc} = 0$, any harmonic spinor is parallel** (Bochner).

REMARK: Due to Bochner's vanishing, **holonomy of Ricci-flat Calabi-Yau manifold lies in $SU(n)$**, and **holonomy of Ricci-flat holomorphically symplectic manifold lies in $Sp(n)$** (a group of complex unitary matrices preserving a complex-linear symplectic form).

Exercise 1: Let (M, ∇) be a manifold with holonomy $\mathrm{Sp}(n)$. Prove that **all parallel $(p, 0)$-forms on M are powers of the holomorphic symplectic form.**

Exercise 2: Let (M, ∇) be a manifold with holonomy $SU(n)$. Prove that **any holomorphic $(p, 0)$-form on M is a parallel section of the canonical bundle.**

Holomorphic Euler characteristic

DEFINITION: A holomorphic Euler characteristic $\chi(M)$ of a Kähler manifold is a sum $\sum(-1)^p \dim H^{p,0}(M)$.

THEOREM: (Riemann-Roch-Hirzebruch) For an n-fold, $\chi(M)$ **can be expressed as a polynomial expressions of the Chern classes,** $\chi(M) = td_n$ where td_n is an n-th component of the Todd polynomial,

$$td(M) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + c_1c_3 + 3c_2^22 - c_4) + ...$$

REMARK: The Chern classes are obtained as polynomial expression of the curvature (Chern-Weil). Therefore $\chi(\tilde{M}) = p\chi(M)$ **for any unramified p-fold covering $\tilde{M} \longrightarrow M$.**

REMARK: Bochner's vanishing and exercises 1-2 imply:

1. When $\mathcal{H}\mathrm{ol}(M) = SU(n)$, we have $\dim H^{p,0}(M) = 1$ **for $p = 1, n$, and 0 otherwise.** In this case, $\chi(M) = 2$ for even n and 0 for odd.

2. When $\mathcal{H}\mathrm{ol}(M) = Sp(n)$,we have $\dim H^{p,0}(M) = 1$ **for even p $0 \leqslant p \leqslant 2n$, and 0 otherwise.** In this case, $\chi(M) = n + 1$.

COROLLARY: $\pi_1(M) = 0$ if $\mathcal{H}\mathrm{ol}(M) = Sp(n)$, or $\mathcal{H}\mathrm{ol}(M) = SU(2n)$. If $\mathcal{H}\mathrm{ol}(M) = SU(2n + 1)$, $\pi_1(M)$ is finite.

The Hodge diamond:

$$
\begin{array}{ccccccc}
 & & & H^{n,n} & & & \\
 & & H^{n,n-1} & & H^{n-1,n} & & \\
 & H^{n,n-2} & & H^{n-1,n-1} & & H^{n-2,n} & \\
\vdots & \vdots & & \vdots & & \vdots & \vdots \\
 & H^{2,0} & & H^{1,1} & & H^{0,2} & \\
 & & H^{1,0} & & H^{0,1} & & \\
 & & & H^{0,0} & & &
\end{array}
$$