Complex algebraic geometry (Hodge theory), exam

Each student receives a random selection of test problems (the output of the randomizer is printed on a separate sheet). The number of the problems is $\max(5, 15 - r)$, where $r = [t/10]$ and t is the total score for the handouts. Each problem is worth 1, 2 or 3 points. The final score for the term is $s = p + r$, where p is the total number of points for the exam. The exam is oral.

1 Differential operators

Exercise 1.1 (2 points). Let M be a compact Riemann manifold and $d + d^* : \Lambda^{\mathsf{even}}(M) \longrightarrow \Lambda^{\mathsf{odd}}(M)$ sum of de Rham differential and its adjoint. Prove that $d + d^*$ is elliptic, and its index is the Euler characteristic of M.

Exercise 1.2 (2 points). Let M be a compact Riemannian manifold and $D : \ C^\infty M \longrightarrow C^\infty M$ a second order elliptic operator. Prove that the index of Δ vanishes.

Exercise 1.3. Let (M, ω, I) be a complex Hermitian manifold, $L(\eta) := \eta \wedge \omega$, $\Lambda := L^*$ the Hodge operators, and d, $d^c := IdI^{-1}$ the corresponding differentials. Consider the operator $\Delta_\omega := d\delta + \delta d$, where $\delta := [d^c, \Lambda]$.

a. Prove that Δ_ω commutes with d and d^c.

b. (2 points) Prove that $\Delta_\omega : \ \Lambda^*(M) \longrightarrow \Lambda^*(M)$ cannot be surjective if M is compact.

Exercise 1.4. Construct an elliptic operator $D : \ F \longrightarrow G$ of order 3 or show that such operators don't exist.

Exercise 1.5. Consider the standard action of $SO(n+1)$ on S^n, and let D be an $SO(n+1)$-invariant second order differential operator. Prove that $D(f) = af + b\Delta(f)$, where Δ is the usual Laplacian associated with the standard metric, and $a, b \in \mathbb{R}$.

Exercise 1.6. Let M be a n-dimensional manifold, $n > 1$. Prove that the set S of order i elliptic operators on $C^\infty M$ is empty for any odd i. Prove that for an even i the set S has two connected components which are convex.

Exercise 1.7. Let f be a smooth function on a compact Riemannian manifold, such that $\Delta(f) = \lambda f$, where $\lambda \in C^\infty M$ is a negative function. Prove that $f = 0$.

Exercise 1.8 (3 points). Let $f \in C^\infty(S^{n-1})$ be an eigenvector of the Laplacian operator on a sphere $S^{n-1} \subset \mathbb{R}^n$ with the usual Riemannian metric. Prove that f can be expressed as a polynomial of coordinate functions on $\mathbb{R}^n$.

Exercise 1.9. Let $V = C^0([0,1])$ be the space of continuous functions on $[0,1]$ with the uniform topology, and $K \subset V$ the space of smooth functions $f : \ [0,1] \longrightarrow \mathbb{R}$ with $|f'| < 1$. Prove that the closure of K is the set of 1-Lipschitz maps $[0,1] \longrightarrow \mathbb{R}$.

2 Almost complex manifolds, connections, symplectic structures

Exercise 2.1. Let f be a holomorphic function on an almost complex manifold. Suppose that $|f|$ is constant. Prove that f is constant.

Exercise 2.2 (3 points). Let (M,ω) be a symplectic manifold. Find a torsion-free connection ∇ such that $\nabla(\omega)=0$.

Exercise 2.3 (2 points). Let ω be a non-degenerate 2-form on a Riemannian manifold, and ∇ its Levi-Civita connection. Assume that $\nabla(\omega)=0$. Prove that M admits a complex structure I such that $\nabla(I)=0$.

Exercise 2.4. Let M be a complex manifold. Construct a torsion-free connection ∇ such that $\nabla(I)=0$.

Exercise 2.5 (2 points). Let (M,I) be an almost complex manifold, and $N:\ \Lambda^2 T^{1,0}M \longrightarrow T^{0,1}M$ its Nijenhuis tensor. Assume that N is surjective. Prove that any holomorphic function on (M,I) is constant.

Exercise 2.6 (2 points). Let V be a 4-dimensional vector space equipped with a scalar product. Construct a natural homeomorphism between $S^2 \coprod S^2$ and the space of all orthogonal complex structures on V.

Exercise 2.7. Construct a G-invariant Hermitian structure on G/H and prove that it is Kähler.

a. (3 points) $G=SO(2n)$, $H=U(n)$.

b. (2 points) $G=U(p,q)$, $H=U(p)\times U(q)$.

c. (2 points) $G=U(p+q)$, $H=U(p)\times U(q)$.

Exercise 2.8 (2 points). Let G be a compact, connected Lie group with a left invariant complex structure and a left invariant Kähler metric. Prove that G is commutative.

Exercise 2.9 (2 points). Let ω be a non-degenerate 2-form on a real manifold M. Prove that there exists an almost complex Hermitian structure such that ω is its Hermitian form.

3 Hodge theory and its applications

Exercise 3.1. Let ω be a non-degenerate 2-form on a $2n$-dimensional smooth manifold, and $d(\omega^k)=0$ for some k satisfying $0<k<n-1$. Prove that $d\omega=0$.

Exercise 3.2. Bi-invariant forms on Lie groups are forms which are invariant under the left and right group action. Let G be a compact Lie group equipped with a bi-invariant metric.

a. (2 points) Prove that all bi-invariant differential forms on G are harmonic.

b. Prove that all harmonic forms are bi-invariant.

Exercise 3.3. Let M be a closed ball in $\mathbb{R}^n$ with a Riemannian metric g which smoothly extends to its boundary, and $\alpha \in \Lambda^k(M)$ a differential form, also smoothly extending to its boundary. Prove that $\alpha \in \operatorname{im} \Delta$, where Δ is the Laplace operator associated with g.

Exercise 3.4 (2 points). Let M be a compact Kähler manifold, d, $d^c := IdI^{-1}$ the usual differential, and $\alpha \in \ker dd^c$. Prove that for any closed (p,q)-form β one has $\int_M \alpha \wedge d\alpha \wedge \beta = 0$.

Exercise 3.5. Let (M, ω) be a compact Kähler manifold, and $\phi \in C^\infty M$ a solution of the Monge-Ampere equation $(\omega + dd^c\phi)^n = e^f \omega^n$, where $f \in C^\infty(M)$. Prove that $\omega + dd^c\phi$ is also a Kähler form.

Exercise 3.6. Let F be an exact holomorphic n-form on an n-dimensional compact complex manifold. Prove that $F = 0$.

Exercise 3.7. Let M be a compact complex manifold, $\dim_{\mathbb{C}} M = 2$. Prove that all holomorphic forms on M are closed.

Exercise 3.8. Let θ be a closed holomorphic 1-form on a simply connected compact complex manifold (not necessarily Kähler). Prove that $\theta = 0$.

Exercise 3.9 (2 points). Let η be a closed (1,1)-form with compact support on $\mathbb{C}^n$, where $n > 1$. Prove that $\eta = dd^c f$, where f is a smooth function on $\mathbb{C}^n$ with compact support.

4 Geometry and topology of Kähler manifolds

Exercise 4.1 (2 points). Let $M = CP^4 \times \mathbb{C}P^4 \times \mathbb{C}P^4$. Prove that M does not admit a Kähler structure with non-standard orientation.

Exercise 4.2. Let M be a compact Kähler manifold, $\dim_{\mathbb{C}} M = 4$. Prove that M does not admit a Kähler structure with opposite orientation or find a counterexample.

Exercise 4.3 (2 points). Let M be a compact complex manifold, and $\pi_1(M) \cong \Gamma$ where Γ is a group of upper triangular integer matrices 4x4 with 1 on diagonal. Prove that M does not admit a Kähler structure.

Exercise 4.4 (3 points). Let $M = \mathbb{C}P^2 \sharp \mathbb{C}P^2$ be a connected sum of $\mathbb{C}P^2$ with itself. Prove that M does not admit a Kähler structure.

Exercise 4.5. For any given $n > 2$ find a $2n$-dimensional simply connected manifold with $b_{2i} \neq 0$, $i = 0, 1, ..., n$ not admitting a symplectic structure.

Exercise 4.6 (2 points). Let M be a compact, non-projective Kähler manifold, $\dim H^{2,0}(M) = 1$, and $\phi : M \longrightarrow M$ a holomorphic involution without fixed points. Prove that ϕ acts trivially on $H^{2,0}(M)$.

Exercise 4.7 (3 points). Let M be a compact, non-projective Kähler manifold, $\dim H^{2,0}(M) = 1$, and Ω a generator of $H^{2,0}(M)$. Consider a submanifold $Z \subset M$ such that $\Omega\big|_Z = 0$. Prove that Z is projective.

Exercise 4.8. Let M be a compact Kähler manifold, $H^{1,1}(M)$ one-dimensional, and $\phi: M \longrightarrow M$ a holomorphic automorphism. Prove that ϕ acts trivially on $H^{1,1}(M)$.

Exercise 4.9 (2 points). Let M be a projective manifold, and $\phi: M \longrightarrow M$ an automorphism. Prove that there exists a ϕ-invariant Kähler metric or find a counterexample.

5 Line bundles and plurisubharmonic functions

Exercise 5.1 (2 points). Let M be a compact complex surface (manifold of complex dimension 2), $\pi: M \longrightarrow S$ a holomorphic map to a curve, and C a smooth fiber of π. Prove that $\deg K_M\big|_C = 2g-2$, where K_M is the canonical bundle of M and g the genus of C.

Exercise 5.2 (2 points). Let f_1, f_2 be holomorphic functions on a Kähler manifold without common zeros. Prove that the function $\Delta(\log(|f_1|^2 + |f_2|^2))$ is non-negative.

Exercise 5.3 (2 points). Let B be a non-trivial line bundle on a compact complex manifold, and h a metric on B with negative curvature of the Chern connection. Prove that B has no non-zero holomorphic sections.

Exercise 5.4 (2 points). Let B be a Hermitian line bundle with positive curvature, f a holomorphic section of B, and Z the set of its zeros. Prove that $dd^c(|f|^{-1})$ is a Kähler form.

Exercise 5.5 (2 points). Let f be a holomorphic function on a Kähler manifold. Prove that the function $\Delta|f|^2$ is non-negative.

Exercise 5.6. Let f be a smooth function with compact support on an n-dimensional Kähler manifold (M, ω). Prove that the integral $\int_M f \wedge dd^c f \wedge \omega^{n-1}$ is non-negative, and vanishes only when $f = \mathsf{const}$.

Exercise 5.7. (local dd^c-lemma) Let η be a closed (1,1)-form on $\mathbb{C}^n$. Prove that $\eta = dd^c\alpha$.

Definition 5.1. A connection is called **compatible with the holomorphic structure** if $\nabla^{0,1} = \bar\partial$.

Exercise 5.8. Let B be a line bundle, and ∇ a connection such that $\nabla(b)$ is holomorphic for any holomorphic section $b \in \Gamma(U, B)$ of B. Prove that the curvature of ∇ is a $(2,0)$-form.

Exercise 5.9. Let M be a complex manifold, ∇ a torsion-free connection preserving I, and ϕ a real function Prove that the 2-form $\operatorname{Hess}(\phi) := \nabla^2(\phi)$ is symmetric. Assume that $\operatorname{Hess}(\phi)$ is positive definite. Prove that $dd^c\phi$ is a Kähler form on M.

Exercise 5.10. Let Pic be the group of holomorphic line bundles on a compact Kähler manifold, with the group structure defined by tensor multiplication. Prove that the conected component of Pic is a compact torus.

Exercise 5.11 (2 points). Prove that a line bundle of degree 0 on a complex curve. admits a Hermitian metric with flat Chern connection.