Hodge theory handout 1: Sheaves

Rules: You may choose to solve only "hard" exercises (marked with !, * and **) or "ordinary" ones (marked with ! or unmarked), or both, if you want to have extra stuff to work. To have a perfect score, a student must obtain (in average) a score of 10 points per week.

If you have got credit for 2/3 of ordinary problems or 2/3 of "hard" problems, you receive $6t$ points, where t is a number depending on the date when it is done. Passing all "hard" or all "ordinary" problems brings you $10t$ points. Solving of "**" (extra hard) problems is not obligatory, but each such problem gives you a credit for 2 "*" or "!" problems in the "hard" set.

The first 3 weeks after giving a handout, $t = 1.5$, between 21 and 35 days, $t = 1$, and afterwards, $t = 0.7$. The scores are not cumulative, only the best score for each handout counts.

1.1 Sheaves and manifolds

Definition 1.1. Let M be a topological space. **A sheaf** $\mathcal{F}$ on M is a collection of vector spaces $\mathcal{F}(U)$ defined for each open subset $U \subset M$, with the **restriction maps**, which are linear homomorphisms $\mathcal{F}(U) \stackrel{\phi_{U,U'}}{\longrightarrow} \mathcal{F}(U')$, defined for each $U' \subset U$, and satisfying the following conditions.

(A) Composition of restrictions is again a restriction: for any open subsets $U_1 \subset U_2 \subset U_3$, the corresponding restriction maps

$$\mathcal{F}(U_1) \stackrel{\phi_{U_1,U_2}}{\longrightarrow} \mathcal{F}(U_2) \stackrel{\phi_{U_2,U_3}}{\longrightarrow} \mathcal{F}(U_3)$$

give $\phi_{U_1,U_2} \circ \phi_{U_2,U_3} = \phi_{U_1,U_3}$.[1]

(B) Let $U \subset M$ be an open subset, and $\{U_i\}$ a covering of U. For any $f \in \mathcal{F}(U)$ such that all restrictions of f to U_i vanish, one has $f = 0$.

(C) Let $U \subset M$ be an open subset, and $\{U_i\}$ a covering of U. Consider a collection $f_i \in \mathcal{F}(U_i)$ of sections, defined for each U_i, and satisfying

$$f_i\Big|_{U_i \cap U_j} = f_j\Big|_{U_i \cap U_j}$$

for each U_i, U_j. Then there exists $f \in \mathcal{F}(U)$ such that the restriction of f to U_i is f_i.

The space $\mathcal{F}(U)$ is called **the space of sections of the sheaf $\mathcal{F}$ on U**. The restriction maps are often denoted $f \longrightarrow f\Big|_U$

Exercise 1.1. Let M be a topological space equipped with a presheaf $\mathcal{F}$. Let $U \subset M$ be an open subset and $\{U_i\}$ its covering. Define maps $\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$ in appropriate way, and prove that the conditions (B) an (C) are equivalent to the exactness of the sequence

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$$

for all open $U \subset M$, and any covering $\{U_i\}$ of U.

[1] If (A) is satisfied, $\mathcal{F}$ is called **a presheaf**.

Definition 1.2. Let $U \subset V$ be open subsets in M. We write $U \Subset V$ if the closure of U is contained in V.

Exercise 1.2 (!). Let $U \Subset V$ be open subsets in a smooth metrizable manifold. Prove that there exists a smooth function $\Phi_{U,V} \in C^\infty M$ equal to 0 outside of the closure of V and equal to 1 on U.

Exercise 1.3. Show that the following spaces of functions on open subsets of $\mathbb{R}^n$ define presheaves, but not sheaves

a. Space of constant functions.

b. Space of bounded functions.

c. Space of functions vanishing outside of a bounded set.

d. Space of continuous functions with finite $\int |f|$.

Definition 1.3. Let $(M, \mathcal{F})$ be a topological manifold equipped with a sheaf of functions. It is said to be a **smooth manifold** of **class** C^∞ or C^i if every point in $(M, \mathcal{F})$ has an open neighborhood isomorphic to the ringed space $(\mathbb{R}^n, \mathcal{F}')$, where $\mathcal{F}'$ is a ring of functions on $\mathbb{R}^n$ of this class.

Exercise 1.4. State the usual definition of a manifold (with charts and atlases). Prove that this definition is equivalent to the one with charts and atlases.

Definition 1.4. A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of rings, such that all restriction maps are ring homomorphisms. We shall only consider ringed spaces equipped with a sheaf of functions, that is, with a subsheaf $\mathcal{F}$ of the sheaf of all functions on M, closed under multiplication. A **morphism** $(M, \mathcal{F}) \stackrel{\Psi}{\longrightarrow} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \stackrel{\Psi}{\longrightarrow} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $f \circ \Psi$ belongs to the ring $\mathcal{F}\big(\Psi^{-1}(U)\big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

Exercise 1.5. Show that any smooth map of manifolds defines a morphism of the corresponding ringed spaces.

Exercise 1.6. Let $f: M \longrightarrow N$ be a map of manifolds which defines a morphism of the corresponding ringed spaces. Prove that it is smooth.

Exercise 1.7 (*). Describe all morphisms of ringed spaces from $(\mathbb{R}^n, C^{i+1})$ to $(\mathbb{R}^n, C^i)$.

1.2 Sheaf morphisms

Definition 1.5. **A sheaf homomorphism** $\psi : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ is a collection of homomorphisms

$$\psi_U : \mathcal{F}_1(U) \longrightarrow \mathcal{F}_2(U),$$

defined for each $U \subset M$, and commuting with the restriction maps. **A sheaf isomorphism** is a homomorphism $\Psi : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$, for which there exists an homomorphism $\Phi : \mathcal{F}_2 \longrightarrow \mathcal{F}_1$, such thate $\Phi \circ \Psi = \mathsf{Id}$ and $\Psi \circ \Phi = \mathsf{Id}$.

Exercise 1.8. Let $\psi : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ be a sheaf homomorphism.

a. Show that $U \longrightarrow \ker \psi_U$ and $U \longrightarrow \operatorname{coker} \psi_U$ are presheaves.

b. Prove that $U \longrightarrow \ker \psi_U$ is a sheaf (it is called **the kernel** of a homomorphism ψ).

c. (*) Prove that $U \longrightarrow \operatorname{coker} \psi_U$ is not always a sheaf (find a counterexample).

Definition 1.6. **A subsheaf** $\mathcal{F}' \subset \mathcal{F}$ is a sheaf associating to each $U \subset M$ a subspace $\mathcal{F}'(U) \subset \mathcal{F}(U)$.

Exercise 1.9. Find a non-zero sheaf $\mathcal{F}$ on M such that $\mathcal{F}(M) = 0$.

Remark 1.1. Let $\phi : A \longrightarrow B$ be a ring homomorphism, and V a B-module. Then V is equipped with a natural A-module structure: $av := \phi(a)v$.

Definition 1.7. **A sheaf of rings** on a manifold M is a sheaf $\mathcal{F}$ with all the spaces $\mathcal{F}(U)$ equipped with a ring structure, and all restriction maps ring homomorphisms.

Definition 1.8. Let $\mathcal{F}$ be a sheaf of rings on a topological space M, and $\mathcal{B}$ another sheaf. It is called **a sheaf of $\mathcal{F}$-modules** if for all $U \subset M$ the space of sections $\mathcal{B}(U)$ is equipped with a structure of $\mathcal{F}(U)$-module, and for all $U' \subset U$, the restriction map $\mathcal{B}(U) \stackrel{\phi_{U,U'}}{\longrightarrow} \mathcal{B}(U')$ is a homomorphism of $\mathcal{F}(U)$-modules (use Remark 1.1 to obtain a structure of $\mathcal{F}(U)$-module on $\mathcal{B}(U')$).

1.3 Germs

Definition 1.9. **The space of germs**, or **stalk** of a sheaf $\mathcal{F}$ at $x \in M$ is the limit $\varinjlim \mathcal{F}(U)$, where U is the set of all neighbourhoods of x, and the maps are restriction maps. Its elements are called **germs**.

Exercise 1.10. Let $\mathcal{F}$ be a ring sheaf on M. Prove that the space of germs of a sheaf of $\mathcal{F}$-modules is a module over the ring of germs of $\mathcal{F}$ in x.

Exercise 1.11. Let $\mathcal{B}$ be a sheaf with all stalks equal 0. Prove that $\mathcal{B} = 0$.

Exercise 1.12 (!). Find a sheaf $\mathcal{F}$ on M with all germs non-zero, and $\mathcal{F}(M)$ zero.

Definition 1.10. A sheaf is called **soft** if for any finite subset $x_1, ..., x_n` \in M$, the natural restriction map $\mathcal{F}(M) \longrightarrow \oplus_i \mathcal{F}_{x_i}$ from the space of global sections to the space of germs is surjective.

Exercise 1.13 (*). Let $\mathcal{F}$ be a soft sheaf on M, and $U \subset M$ an open subset. Prove that the map $\mathcal{F}(M) \longrightarrow \mathcal{F}(U)$ is always surjective, or find a counterexample.

Exercise 1.14 (!). Let M be a smooth, metrizable manifold, and $\mathcal{F}$ be a sheaf of modules over $C^\infty(M)$. Prove that $\mathcal{F}$ is soft.

Hint. Use a partition of unity.

Definition 1.11. A free sheaf of modules $\mathcal{F}^n$ over a ring sheaf $\mathcal{F}$ is a sheaf such that the space of sections over an open set U is $\mathcal{F}(U)^n$. A sheaf of $\mathcal{F}$-modules is **non-free** if it is not isomorphic to a free sheaf.

Exercise 1.15 (!). Find a subsheaf of modules in $C^\infty M$ which is non-free in the sense of this definition.

Definition 1.12. Locally free sheaf of modules over a sheaf of rings $\mathcal{F}$ is a sheaf of modules $\mathcal{B}$ satisfying the following condition. For each $x \in M$ there exists a neighbourhood $U \ni x$ such that the restriction $\mathcal{B}\big|_U$ is free.

Definition 1.13. A vector bundle on a ringed space $(M, \mathcal{F})$ is a locally free sheaf of $\mathcal{F}$-modules.

Exercise 1.16 (!). Given a smooth manifold M, define the tangent bundle TM, and prove that it is a locally free sheaf of $C^\infty M$-modules.

Exercise 1.17. Prove that the tangent bundle is a free sheaf for the following manifolds.

a. $M = \mathbb{R}$

b. $M = S^1$ (a circle)

c. $M = \mathbb{R}^2/\mathbb{Z}^2$ (a torus)

d. (*) $M = S^3$ (a three-dimensional sphere)

Exercise 1.18 (!). Let B be a vector bundle on a manifold $(M, C^\infty M)$. Prove that B is soft (as a sheaf).

Exercise 1.19 (**). Let B_1, B_2 be vector bundles on (M, C^∞) such that the spaces of sections $B_1(M)$ and $B_2(M)$ are isomorphic as $C^\infty(M)$-modules. Prove that the bundles B_1 and B_2 are isomorphic.

Definition 1.14. Let $\mathcal{F}$ be a sheaf of $C^\infty M$-modules, and $\mathcal{F}_x$ its germ in x. Denote the quotient $\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$ by $\mathcal{F}\big|_x$. This space is called **the fiber** of $\mathcal{F}$ in x. A morphism of sheaves induces a linear map on each of its fibers.

Exercise 1.20 (**). Let $\mathcal{F}$ be a sheaf of $C^\infty M$-modules such that all its fibers $\mathcal{F}\big|_x$ vanish. Prove that $\mathcal{F}$ is zero, or find a counterexample.