Hodge theory 2: Differential operators

Rules: You may choose to solve only "hard" exercises (marked with !, * and **) or "ordinary" ones (marked with ! or unmarked), or both, if you want to have extra stuff to work. To have a perfect score, a student must obtain (in average) a score of 10 points per week.

If you have got credit for 2/3 of ordinary problems or 2/3 of "hard" problems, you receive $6t$ points, where t is a number depending on the date when it is done. Passing all "hard" or all "ordinary" problems brings you $10t$ points. Solving of "**" (extra hard) problems is not obligatory, but each such problem gives you a credit for 2 "*" or "!" problems in the "hard" set.

The first 3 weeks after giving a handout, $t = 1.5$, between 21 and 35 days, $t = 1$, and afterwards, $t = 0.7$. The scores are not cumulative, only the best score for each handout counts.

2.1 Differential operators (after Grothendieck)

Definition 2.1. Let R be a commutative ring over a field k. Given $a \in R$, consider the map $L_a : R \longrightarrow R$ mapping x to ax. Define $\operatorname{Diff}^k(R) \subset \operatorname{Hom}_k(R, R)$ inductively as follows. The $\operatorname{Diff}^0(R)$ is the space of all R-linear maps from R to R, that is, the space of all L_a, $a \in R$. The space $\operatorname{Diff}^k(R)$, $k > 0$ is

$$\operatorname{Diff}^k(R) := \{D \in \operatorname{Hom}_k(R, R) \mid [L_a, D] \in \operatorname{Diff}^{k-1}(R) \quad \forall a \in R.\}$$

The union of all $\operatorname{Diff}^i(R)$ is called **the space of differential operators on** R. Differential operators on the ring $C^\infty M$ is called **differential operators on** M**,** denoted $\operatorname{Diff}^*(M)$.

Exercise 2.1. Let $D^i \in \operatorname{Diff}^i(R)$, $D^j \in \operatorname{Diff}^j(R)$ be differential operators. Prove that the composition $D^i D^j$ lies in $\operatorname{Diff}^{i+j}(R)$.

Hint. Use induction and identity $[v, D^i D^j] = [v, D^i]D^j + D^i[v, D^j]$

Exercise 2.2. Let $D^i \in \operatorname{Diff}^i(R)$, $D^j \in \operatorname{Diff}^j(R)$ be differential operators. Prove that the commutator $[D^i, D^j]$ lies in $\operatorname{Diff}^{i+j-1}(R)$.

Hint. Use induction and Jacobi identity

$$[v, [D^i, D^j]] = [[v, D^i], D^j] + [D^i, [v, D^j]].$$

Definition 2.2. Let R be a k-algebra, and $D : R \longrightarrow A$ a k-linear map from R to an R-module. It is called **a k-derivation**, or just **derivation** if it satisfies the Leibniz rule: $D(xy) = yD(x) + xD(y)$.

Exercise 2.3. a. Prove that $D(k) = 0$ for any k-derivation on a k-algebra (we assume $\operatorname{char} k = 0$).

b. (!) Let R be a finite extension of a field k of characteristic 0. Prove that the space $\operatorname{Der}_k(R, R)$ of derivations vanishes.

Exercise 2.4 (**). Let R be the ring of continuous functions on a manifold M. Prove that $\operatorname{Der}_{\mathbb{R}}(R, R) = 0$, or find a counterexample.

Exercise 2.5 (*). Let $x_1, ..., x_n$ be coordinates on $\mathbb{R}^n$. Prove that any derivation on $C^\infty\mathbb{R}^n$ is written as coordinates as $D(f) = \sum_{i=1}^n f_i \frac{d}{dx_i}$, where $f_i \in C^\infty M$.

Hint. Use the Hadamard lemma and an inclusion $D(I^k) \subset I^{k-1}$ (Exercise 2.8).

Exercise 2.6 (!). Let $D \in \operatorname{Diff}^1(R)$ be a differential operator of first order. Prove that $D - D(1)$ is a derivation of R. Prove that $\operatorname{Diff}^1(R)/\operatorname{Diff}^0(R)$ is isomorphic to the space of derivations of R.

Exercise 2.7. Let $R = k[t]$ be an algebra of polynomials over a field k of characteristic 0, and $D \in \operatorname{Diff}^k(R)$.

a. Prove that D is uniquely determined by its restriction on polynomials of degree $\leqslant k$.

b. (*) Prove that $\operatorname{Diff}^k(R)$ is a free $k[t]$-module, generated by $\tau_0, \tau_1, ...\tau_k$, where τ_i maps all $1, t, t^2, t^3, ...t^k$ except t^i to 0, and t^i to 1.

c. (**) Prove that $\operatorname{Diff}^*(\mathbb{R}[t_1, ..., t_n])$ is an algebra freely generated by generators $t_1, ..., t_n$ and $\frac{d}{dt_1}, ..., \frac{d}{dt_n}$ and relations $[t_i, \frac{d}{dt_j}] = \delta_{ij}$.

Exercise 2.8. Let $I \subset R$ be an ideal, and $D \in \operatorname{Diff}^k(R)$. Prove that $D(I^{k+1}) \subset I$.

Hint. Use induction in k and identity $[D, L_a L_b] = [D, L_a]L_b + L_a[D, L_b]$.

2.2 Differential operators are local

Definition 2.3. Let $f \in \mathcal{F}(M)$ be a section of a sheaf $\mathcal{F}$, and $U \subset M$ the union of all open subsets $V \subset M$ such that $f\Big|_V = 0$. The complement $M \backslash U$ is called **support** of M, denoted $\operatorname{Supp}(f)$.

Exercise 2.9. Prove that $\operatorname{Supp}(f)$ is the set of all $m \in M$ such that the germ of f in m is non-zero.

Definition 2.4. Let $f, g \in C^\infty M$. We say that f **is divisible by** g if $f = f'g$ for some $f' \in C^\infty M$.

Exercise 2.10. Let $K \subset U \subset M$ be subsets of a manifold M, where U is open and K compact.

a. Prove that any function with support in K is divisible by any function which is non-zero in U.

b. Let f be a function on M which is non-zero somewhere in $U \backslash K$. Prove that there exists a function g with $\operatorname{Supp}(g) \supset K$ such that f is not divisible by g.

Hint. If f vanishes in $x \in U$, but support of f contains x, construct g which is equal to f^2 in a neighbourhood of x and supported in $K \subset U$, and show that f is not divisible by g.

Exercise 2.11. Let $D \in \operatorname{Diff}^*(M)$ be a differential operator, and f a function divisible by any power of g. Prove that $D(f)$ is divisible by any power of g.

Definition 2.5. An operator $D: C^\infty M \longrightarrow C^\infty M$ is called **local** if it $\operatorname{Supp}(D(f)) \subset \operatorname{Supp}(f)$ for any function $f \in C^\infty M$.

Exercise 2.12.

a. Let K be a closed subset of M. Prove that $\operatorname{Supp}(f) \subset K$ if and only if f is infinitely divisible by any function g which is non-zero everywhere in K.

b. (!) Prove that any differential operator (in the sense of Grothendieck) is local.

Hint. Use Exercise 2.10 and Exercise 2.11.

2.3 Jet spaces

Definition 2.6. Let M be a smooth manifold, $f \in C^\infty M$ a function, and $x \in M$ a point. Denote by $\mathfrak{m}_x$ the maximal ideal of x, that is, the ideal of all functions vanishing in x. The **k-jet of f in x** is an image of f in $C^\infty M/\mathfrak{m}_x^{k+1}$. **The space of k-jets of functions in x** is $C^\infty M/\mathfrak{m}_x^{k+1}$, and the corresponding vector bundle $\operatorname{Jet}^k(M)$ with the fiber in x equal to $C^\infty M/\mathfrak{m}_x^{k+1}$ is called **the bundle of k-jets**.

Exercise 2.13. Let Δ be the diagonal in $M \times M$, and $J_\Delta \subset C^\infty(M \times M)$ the ideal of all functions vanishing in M. Denote by $\pi: M \times M \longrightarrow M$ the projection to the first component

a. (!) Prove that the cotangent bundle T^*M is isomorphic as a sheaf of $C^\infty M$-modules to J_Δ/J_Δ^2, considered as a sheaf on $\Delta = M$, with $C^\infty M$-action induced by $\pi^* C^\infty M\Big|_\Delta \cong C^\infty M$ (that is, we take the functions which are constant along the second component, and consider J_Δ/J_Δ^2 as a sheaf of modules over such functions).

b. (*) Prove that the bundle of k-jets is identified, in a similar way, with the sheaf $C^\infty(M \times M)/J_\Delta^k$, considered as a sheaf on $\Delta = M$, with $C^\infty M$-action induced by $\pi^* C^\infty M\Big|_\Delta \cong C^\infty M$.

Definition 2.7. For any $f \in C^\infty M$, define **the k-jet of f** as a section of $\operatorname{Jet}^k(M)$ which is equal to the k-jet of f at each $x \in M$.

Exercise 2.14. Let $D \in \operatorname{Diff}^k(M)$ be a differential operator (in the sense of Grothendieck), $f \in C^\infty M$, and $J^k(f)$ its k-jet. Prove that for each $x \in M$, the number $D(f)(x)$ depends linearly on the k-jet of f in x.

Hint. Use Exercise 2.8.

Exercise 2.15 (!). In these assumptions, prove that there exists a $C^\infty M$-linear map $D_J: \operatorname{Jet}^k(M) \longrightarrow C^\infty M$ such that $D(f) = D_J(J^k(f))$.

Hint. Use the previous exercise.

Exercise 2.16 (*). Prove that $\operatorname{Diff}^k(M)$ is a vector bundle. Prove that this vector bundle is dual to $\operatorname{Jet}^k(M)$.

Hint. Use locality, dimension count and the previous exercise.

Exercise 2.17 (*). Prove that all differential operators $D \in \operatorname{Diff}^k(M)$ (in the sense of Grothendieck) can be expressed in local coordinates as

$$D = f_0 + \sum_{i=1}^n f_i \frac{\partial}{\partial x_i} + \sum_{i,j=1}^n f_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i,j,k=1}^n f_{ijk} \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} + ...$$

Hint. Use the previous exercise.

Exercise 2.18 (**). Prove that any local operator on a compact manifold is a differential operator.

2.4 The ring of symbols of differential operators

Definition 2.8. Let R be an associative algebra. **(Increasing) filtration** on R is a collection of subspaces $R_0 \subset R_1 \subset R_2 \subset ...$ such that $R_i R_j \subset R_{i+j}$. The natural product map $(R_k/R_{k-1}) \otimes (R_l/R_{l-1}) \longrightarrow R_{k+l}/R_{k+l-1}$ defines an associative product structure on the space $\bigoplus_{i=0}^\infty R_i/R_{i-1}$. The algebra $\bigoplus_{i=0}^\infty R_i/R_{i-1}$ is called **the associated graded algebra** of this filtration.

Exercise 2.19. Consider the algebra $\operatorname{Diff}^*(R)$ with its filtration by $\operatorname{Diff}^i(M)$. Prove that its associated graded algebra is commutative.

Hint. Use Exercise 2.2.

Definition 2.9. This ring is called **the ring of symbols of differential operators**. For any $D \in \operatorname{Diff}^k(R)$, its representative in $\operatorname{Diff}^k(R)$ is called **the symbol of** D.

Exercise 2.20. Consider sections of TM as differential operators of the first order.

a. Prove that $TM = \operatorname{Diff}^1 M / \operatorname{Diff}^0 M$.

b. Prove that the multiplication in the ring of symbols defines a surjective, $C^\infty M$-linear map $\operatorname{Sym}^k TM \longrightarrow \operatorname{Diff}^k M / \operatorname{Diff}^{k-1} M$.

c. (!) Prove that this map is an isomorphism.