Hodge theory 3: Stone-Weierstrass theorem and Banach spaces

Rules: You may choose to solve only "hard" exercises (marked with !, * and **) or "ordinary" ones (marked with ! or unmarked), or both, if you want to have extra stuff to work. To have a perfect score, a student must obtain (in average) a score of 10 points per week.

If you have got credit for 2/3 of ordinary problems or 2/3 of "hard" problems, you receive $6t$ points, where t is a number depending on the date when it is done. Passing all "hard" or all "ordinary" problems brings you $10t$ points. Solving of "**" (extra hard) problems is not obligatory, but each such problem gives you a credit for 2 "*" or "!" problems in the "hard" set.

The first 3 weeks after giving a handout, $t = 1.5$, between 21 and 35 days, $t = 1$, and afterwards, $t = 0.7$. The scores are not cumulative, only the best score for each handout counts.

3.1 Weierstrass approximation theorem

Definition 3.1. Let M be a topological space, and $\|f\| := \sup_M |f|$ **the sup-norm on functions**. C^0-topology on the space $C^0(M)$ of bounded continuous functions is topology defined by the sup-norm.

Exercise 3.1. Prove that $C^0 M$ with the metric defined by the sup-norm is a complete metric space.

Exercise 3.2. ("Dini's theorem")
Let $\{f_i\}$ be a sequence of continuous functions on a compact space M, and suppose that $f_i(t) \geqslant f_{i-1}(t)$ for all t and i. Suppose that $\lim_i f_i(t) = f(t)$ for some continuous function f. Prove that the sequence $\{f_i(t)\}$ converges to $f(t)$ uniformly.

Exercise 3.3. Consider the sequence P_i, $i = 0, 1, 2, ...$ of polynomials on $[0, 1]$ determined recursively as follows: $P_0(t) = 0$, and $P_i(t) = P_{i-1}(t) + \frac{1}{2}(t - P_{i-1}(t)^2)$. For all $t \in [0, 1]$ and all $i = 1, 2, ...,$, prove the following.

a. Prove that $0 \leqslant P_i(t) \leqslant \sqrt{t}$.

b. Prove that $P_i(t) \geqslant P_{i-1}(t)$.

c. Prove that $\{P_i(t)\}$ converges pointwisely to $\sqrt{t}$ on $[0, 1]$.

d. Prove that $\{P_i(t)\}$ converges uniformly to $\sqrt{t}$ on $[0, 1]$

e. Prove that $Q_i(t) := P_i(t^2)$ converges uniformly to $|t|$ on $[-1, 1]$.

Exercise 3.4. Let $F(t)$ be a piecewise linear, continuous function on $[a, b] \subset \mathbb{R}$. Prove that $F(t)$ can be expressed as a sum $\sum_{i=0}^n \alpha_i |x - c_i|$ for some α_i, c_i.

Exercise 3.5. Prove that any piecewise linear, continuous function on $[a, b] \subset \mathbb{R}$ can be obtained as a uniform limit of polynomials.

Exercise 3.6 (!). (Weierstrass approximation theorem)
Prove that any continuous function on $[a, b] \subset \mathbb{R}$ admits a uniform approximation by polynomials.

Remark 3.1. This particular proof of Weierstrass approximation is due to Lebesgue.

3.2 Stone-Weierstrass approximation theorem

From now on we assume that M is compact, Hausdorff topological space.

Definition 3.2. Let $A \subset C^0 M$ be a subspace in the space of continuous functions. We say that A **separates the points** of M if for every points $x \neq y \in M$, there exists $f \in A$ such that $f(x) \neq f(y)$.

Exercise 3.7. Let $A \subset C^0 M$ be an $\mathbb{R}$-subalgebra, and $\bar{A}$ its closure in C^0-topology.

a. Prove that for any $a \in A$, the function $|a|$ belongs to $\bar{A}$.

b. Prove that for any $a, b \in A$, the function $\min(a, b)$ belongs to $\bar{A}$.

Hint. Use Exercise 3.3.

Exercise 3.8. Let $A \subset C^0 M$ be a subring separating points, $\bar{A}$ its closure, and $U \ni x$ a neighbourhood of $x \in M$. Prove that for any $\varepsilon > 0$ there exists $a \in \bar{A}$ taking values in $[0, 1]$ such that $a(x) = 1$ and $a\Big|_{M \backslash U} < \varepsilon$.

Hint. Find a finite covering of the compact $M \backslash U$ by open sets U_i and functions $f_i \in \bar{A}$ taking values in $[0, 1]$ such that $f_i(x) = 1$ and $f_i\Big|_{U\ i} < \varepsilon$, and put $a := \min_i(f_i)$.

Exercise 3.9. Let $A \subset C^0 M$ be a subring separating points, $\bar{A}$ its closure, and $f \in C^0(M)$ any function. Prove that for all $x \in M$ there exists a function $f_x \in \bar{A}$ such that $f_x \leqslant f$ and $f_x(x) > f(x) - \varepsilon$.

Hint. Use the previous exercise.

Exercise 3.10 (!). (Stone-Weierstrass theorem)
Let $A \subset C^0 M$ be a subring separating points, and $\bar{A}$ its closure. Prove that $\bar{A} = C^0 M$.

Hint. Use the previous exercise and find a neighbourhood U_x and a function $f_x \leqslant f$ such that $(f_x + \varepsilon)\Big|_{U_x} > f\Big|_{U_x}$. Find a finite covering $\{U_{x_i}\}$ by such U_x, such that $f \geqslant \max_i f_{x_i} > f - \varepsilon$.

3.3 Banach spaces

Definition 3.3. A non-negative real-valued function $v \longrightarrow |v|$ on a vector space is called **norm** if

- $|v| = 0$ if and only if $v = 0$
- For each $r \in \mathbb{R}$, $|rv| = |r||v|$.
- $|v_1 + v_2| \leqslant |v_1| + |v_2|$.

Clearly, norm defines a metric on V, with $d(x, y) := |x - y|$. A norm is **complete** if this metric is complete.

Definition 3.4. A complete normed topological vector space is called **Banach space**.

Definition 3.5. Let V_1, V_2 be vector spaces equipped with a norm. **Norm** ("operator norm") of a linear operator $E: V_1 \longrightarrow V_2$ is the number $\|E\| := \sup_{v \neq 0} \frac{|E(v)|}{|v|}$. An operator with finite norm is called **bounded**.

Exercise 3.11 (!). Show that E is continuous if and only if it is bounded.

Exercise 3.12. Prove that $\|E\|$ defines a norm on the space of bounded operators $E: V_1 \longrightarrow V_2$.

Exercise 3.13 (!). Suppose that V_1, V_2 are Banach spaces, and $\operatorname{Hom}(V_1, V_2)$ the space of bounded operators equipped with the operator norm. Prove that $\operatorname{Hom}(V_1, V_2)$ is a Banach space.

Remark 3.2. From now on, all linear operators on topological vector spaces are assumed continuous, unless stipulated otherwise.

Definition 3.6. Let H be an infinite-dimensional vector space equipped with a positive-definite scalar product. We say that H is a **Hilbert space** if it is complete and contains a countable dense set.

Exercise 3.14 (!). Let g denote the scalar product on a Hilbert space H. Prove that g defines an isomorphism from H to H^*, whete H^* denotes the space of continuous functionals.

Hint. Let x_i be the orthonormal basis in H. Use the dual basis to write any form as $\lambda = \sum_i \lambda_i e_i$. Prove that $\frac{|\lambda(x)|^2}{|x|^2} = \sum |\lambda_i|^2$.

Exercise 3.15 (*). Prove that a sphere in a Hilbert space is contractible.

Exercise 3.16 (*). Consider the group $GL(H)$ of linear automorphisms of a Hilbert space, taken with the norm topology. Prove that it is contractible.

Exercise 3.17 (**). Consider the group $O(H)$ of linear isometries of a Hilbert space, taken with the norm topology. Prove that it is contractible.

Exercise 3.18. a. Let H be a Hilbert space. Prove that the closure of the unit ball is non-compact.

b. (*) Let V be an infinite-dimensional Banach space. Prove that the closure of the unit ball is non-compact.

Exercise 3.19 (**). Let $E;\ V_1 \longrightarrow V_2$ be a bijective, continuous linear operator on Hilbert spaces. Prove that E^{-1} is bounded.

Definition 3.7. Let V be a vector space, and $|\ |_1$, $|\ |_2$ - norms on V. We say that these norms are **equivalent** if the identity operator $(H, |\ |_1) \longrightarrow (V, |\ |_2)$ is a homeomorphism.

Exercise 3.20 (!). Prove that $|\ |_1$, $|\ |_2$ are equivalent if and only if there exists constant $C > 1$ such that for all $v \in V$ one has $C^{-1}|v|_1 \leqslant |v|_2 \leqslant C|v|_1$.

Exercise 3.21. Prove that on a finite-dimensional space all norms are equivalent.