Hodge theory 4: Compact operators

Rules: You may choose to solve only "hard" exercises (marked with !, * and **) or "ordinary" ones (marked with ! or unmarked), or both, if you want to have extra stuff to work. To have a perfect score, a student must obtain (in average) a score of 10 points per week.

If you have got credit for 2/3 of ordinary problems or 2/3 of "hard" problems, you receive $6t$ points, where t is a number depending on the date when it is done. Passing all "hard" or all "ordinary" problems brings you $10t$ points. Solving of "**" (extra hard) problems is not obligatory, but each such problem gives you a credit for 2 "*" or "!" problems in the "hard" set.

The first 3 weeks after giving a handout, $t = 1.5$, between 21 and 35 days, $t = 1$, and afterwards, $t = 0.7$. The scores are not cumulative, only the best score for each handout counts.

4.1 Compact operators

Definition 4.1. A set is called **precompact** if its closure is compact. A subset B of a topological vector space is called **bounded** if for any neighbourhood U of 0, there exists a constant $\lambda > 0$ such that λU contains B. An operator on topological vector spaces is called **compact** if the image of any bounded set is precompact.

Exercise 4.1. Prove that an open set in a Hilbert space is never precompact.

Exercise 4.2 (*). (Riesz theorem) Prove that an open set in a normed infinite-dimensional vector space is never precompact.

Exercise 4.3 (**). Construct an infinite-dimensional locally convex topological vector space H such that any bounded subset of H is precompact.

Definition 4.2. Let H be a Hilbert space, and $e_1, ..., e_n, ...$ an orthonormal basis. Then any point in H can be expressed as $\sum \alpha_i e_i$ with $\alpha_i \in \mathbb{R}$ and $\sum |\alpha_i|^2 < \infty$. Let $\{x_i\}$ be a sequence of positive numbers with $\sum x_i^2 < 0$ **The Hilbert cube** is the set of all vectors $\sum \alpha_i e_i \in H$ satisfying $|\alpha_i| \leqslant x_i$.

Exercise 4.4. Prove that the Hilbert cube is compact.

Exercise 4.5. Let $K: H \longrightarrow H_1$ be a operator on Hilbert spaces.

a. Suppose that K is compact. Prove that for any $\varepsilon > 0$ there exists a subspace $W \subset H$, closed and of finite codimension, such that the operator norm of the restriction $K\big|_W$ satisfies $\big\|K\big|_W\big\| < \varepsilon$.

b. (*) Prove the converse: any operator with this property is compact.

4.2 Weak topology

Definition 4.3. Let $x_i \in H$ be a sequence of points in a Hilbert space H. We say that x_i **weakly converges** to $x \in H$ if for any $z \in H^*$ one has $\lim_i \langle x_i, z\rangle = \langle x, z\rangle$. **Weak topology** on H is topology with subbase of open sets given by $\lambda^{-1}(]a, b[)$, where λ is any continuous functional.

Exercise 4.6. Prove that an open ball in a Hilbert space is not open in the weak topology.

Exercise 4.7. Let $y(i) = \alpha_j(i)e_j$ be a sequence of points in a unit ball in the Hilbert space with orthonormal basis e_i. Prove that $y(i)$ converges to $y = \sum \alpha_i y_i$ in weak topology if and only if $\lim_i \alpha_j(i) = \alpha_j$ for all j.

Exercise 4.8. Prove that the closure of the unit ball in a Hilbert space is compact in weak topology.

Exercise 4.9 (!). Let $A : H \longrightarrow H$ be a continuous operator on a Hilbert space. Prove that A is compact if and only if it maps weakly converging sequences to converging sequences, and their weak limits to limits.

Exercise 4.10. Let $K : H \longrightarrow H$ be a compact operator on a Hilbert space, and B a closure of the unit ball. Prove that $K(B)$ is compact.

4.3 Von Neumann spectral theorem

Definition 4.4. **Spectrum** $\operatorname{Spec}(A)$ of a continuous operator $A : H \longrightarrow H$ on a Banach space is the set of all $\lambda \in \mathbb{C}$ such that $A - \lambda \operatorname{Id}$ is not invertible.

Exercise 4.11 (!). Let A be a continuous operator $A : H \longrightarrow H$ on a Banach space. Prove that its spectrum is closed in $\mathbb{C}$.

Exercise 4.12 (*). Prove that the spectrum of any operator is non-empty.

Exercise 4.13. Find an operator on a Hilbert space with spectrum a unit circle in $\mathbb{C}$.

Exercise 4.14 (**). Let ε be a positive number, and K a compact operator on a Hilbert space. Prove that the set $\{\lambda \in \operatorname{Spec} K \mid |\lambda| > \varepsilon\}$ is finite.

Definition 4.5. Let $K: H \longrightarrow H$ be an operator on a Banach space, and $H_\lambda := \bigcup_n \operatorname{ker}(K - \lambda \operatorname{Id}_H)^n$. The space H_λ is called **the root space** of H.

Exercise 4.15. Let $K: H \longrightarrow H$ be a compact operator on Hilbert spaces, and H_λ its root spaces. Prove that

a. for any $\lambda \neq 0$, the space H_λ is finite-dimensional.

b. (*) Prove that $\operatorname{Spec}(K)$ is countable.

c. (**) Prove that H is the closure of $\bigoplus_{\lambda \in \operatorname{Spec}(K)} H_\lambda$, or find a counterexample.

Exercise 4.16 (**). Construct an injective compact operator K with $\operatorname{Spec}(K) = \{0\}$, or find a counterexample.

Definition 4.6. An operator $A: H \longrightarrow H$ on a Hilbert space is called **self-adjoint** if $A^* = A$.

Exercise 4.17 (!). Let $A: H \longrightarrow H$ be a compact self-adjoint operator. Prove that there exists $x \in H$ such that $\frac{|A(x)|}{|x|} = \sup_h \frac{|A(h)|}{|h|}$.

Hint. Use the weak compactness of the closed ball.

Exercise 4.18. Let $A: H \longrightarrow H$ be a self-adjoint operator, and z a unit vector such that $\frac{|A(z)|}{|z|} = \sup_h \frac{|A(h)|}{|h|}$.

a. Prove that $\|A^2\| = \|A\|^2$, and $\frac{|A^2(z)|}{|z|} = \sup_h \frac{|A^2(h)|}{|h|}$.

b. (!) Prove that z is an eigenvector for A^2.

Hint. Prove that $g(A^2(z), z) = |z||A^2(z)|\cos\phi$, where ϕ is an angle between z and $A^2(z)$.

Exercise 4.19 (!). Let $A: H \longrightarrow H$ be a compact self-adjoint operator, and z a unit vector such that $\frac{|A(z)|}{|z|} = \sup_h \frac{|A(h)|}{|h|}$. Prove that $z^\perp$ is A^2-invariant, and use this to show that A^2 is diagonalizable.

Hint. Use the previous exercise.

Exercise 4.20 (!). Let $A: H \longrightarrow H$ be a compact self-adjoint operator on a Hilbert space. Prove that A is diagonalizable in an orthonormal basis.

Hint. Use the previous exercise.