Hodge theory 5: Fredholm operators

Rules: You may choose to solve only "hard" exercises (marked with !, * and **) or "ordinary" ones (marked with ! or unmarked), or both, if you want to have extra stuff to work. To have a perfect score, a student must obtain (in average) a score of 10 points per week.

If you have got credit for 2/3 of ordinary problems or 2/3 of "hard" problems, you receive $6t$ points, where t is a number depending on the date when it is done. Passing all "hard" or all "ordinary" problems brings you $10t$ points. Solving of "**" (extra hard) problems is not obligatory, but each such problem gives you a credit for 2 "*" or "!" problems in the "hard" set.

The first 3 weeks after giving a handout, $t = 1.5$, between 21 and 35 days, $t = 1$, and afterwards, $t = 0.7$. The scores are not cumulative, only the best score for each handout counts.

5.1 Banach inverse map theorem

Exercise 5.1. Let $E: V_1 \longrightarrow V_2$ be a bijective linear map of normed spaces. Prove that E is a homeomphism if and only if there exist a constant $C > 0$ such that $C^{-1}|v| < |E(v)| < C|v|$ for any $v \in V_1$.

Exercise 5.2. Let $K: H_1 \longrightarrow H_2$ be a compact operator with zero kernel; consider its action on the projective spaces $K_{\mathbb{P}}: \mathbb{P}H_1 \longrightarrow \mathbb{P}H_2$. Prove that the closure of the image of an open ball cannot contain a neighbourhood of 0.

Exercise 5.3 (!). Prove that a compact operator of Hilbert spaces is never surjective.

Exercise 5.4. Let H be a Hilbert space, and H^* the space of continuous linear functionals on H. Consider the map $\tau: H \longrightarrow H^*$ defined by the scalar product g, $x \longrightarrow g(x, \cdot)$. Prove that τ is an isomorphism.

Exercise 5.5. Let $F: H_1 \longrightarrow H_2$ be a continuous operator on Hilbert spaces, and $F^*: H_2 \longrightarrow H_1$ an operator which satisfies $(x, F(h)) = (F^*(x), h)$ for any $x \in H_2, h \in H_1$.

a. (!) Prove that such an operator always exists.

b. Prove that it is uniquely defined.

c. Prove that $F^{**} = F$.

d. (*) Prove that F^* is compact if and only if F is compact.

Definition 5.1. In this case F^* is called **adjoint** to F.

Exercise 5.6. Let $A: \quad H \longrightarrow H_1$ be a continuous operator on Hilbert spaces. Prove that $\ker A^*$ is equal to $(\operatorname{im} A)^\perp$.

Exercise 5.7. Let $A: \ V \longrightarrow W$ be a continuous operator on Hilbert spaces, and $V = V_1 \oplus V_2$ be an orthogonal decomposition. Denote by A_1 the map which is equal to A on V_1 and 0 on V_2 and A_2 the map which is equal to A on V_2 and 0 on V_1.

a. Prove that $\ker A_i^* = A(V_i)^\perp$.

b. Prove that $A(V_1)^\perp \cap A(V_2)^\perp = \ker A^*$. Prove that $(A^*)^{-1}(V_1) = A(V_2)^\perp$.

c. Suppose that A is bijective. Prove that A^* is injective, and $(A^*)^{-1}$ applied to the decomposition $V = V_1 \oplus V_2$ gives a decomposition $W = A(V_1)^\perp \oplus A(V_2)^\perp$.

d. (!) Suppose that A is bijective. Prove that $A(V_1)^\perp + A(V_2)^\perp = W$ and $A(V_1)^\perp \cap A(V_2)^\perp = 0$. Prove that the spaces $A(V_i)$ are closed in W.

Exercise 5.8 (!). Let $A: \ V \longrightarrow W$ be a continuous bijective operator on Hilbert spaces. Prove that either A^{-1} is continuous or there exists a subspace $V_1 \subset V$ such that $A\Big|_{V_1}$ is compact.

Exercise 5.9 (!). Let A be a continuous bijective operator on Hilbert spaces. Prove that A^{-1} is continuous.

Hint. Use Exercises 5.3, 5.7 and 5.8.

Exercise 5.10 (**). Let $A: \ V_1 \longrightarrow V_2$ be a bijective, continuous map of Banach spaces. Prove that A^{-1} is continuous.

5.2 Fredholm operators

Definition 5.2. A continuous map $F: \ H_1 \longrightarrow H_2$ of Hilbert spaces is called **Fredholm** if its kernel is finite-dimensional, image closed, and cokernel finite-dimensional.

Exercise 5.11. Let $F: \ H_1 \longrightarrow H_2$ be a Fredholm operator. Prove that F induces a homeomorphism from $H_1/\ker F$ to $\operatorname{im} F$.

Definition 5.3. An operator on Hilbert spaces **has finite rank** if its image is finite-dimensional.

Exercise 5.12. Let $F: H_1 \longrightarrow H_2$ be a Fredholm operator.

a. Prove that there exists a Fredholm operator $F_1: H_2 \longrightarrow H_1$ such that $\mathsf{Id}_{H_1} - FF_1$ has finite rank.

b. Prove that in this case $\mathsf{Id}_{H_2} - F_1F$ has finite rank.

Exercise 5.13. Let $K: H \longrightarrow H$ be a compact operator.

a. Prove that the image of $\mathsf{Id}_H + K$ is closed.

b. (!) Prove that $\mathsf{Id}_H + K$ is Fredholm.

5.3 Calkin algebra

Definition 5.4. Let R be a k-algebra. **Two-sided ideal** in R is a subspace $I \subset R$ such that $IR = I$ and $RI = I$.

Exercise 5.14. Let A be the algebra of continuous operators $\operatorname{Hom}(H, H)$ from a Hilbert space to itself. Prove that the space K of compact operators is a two-sided ideal in A.

Definition 5.5. The quotient algebra $\operatorname{Hom}(H, H)/K$ is called **the Calkin algebra** of H.

Exercise 5.15 (**). Prove that the Calkin algebra admits a continuous homeomorphism to a closed subalgebra of $\operatorname{Hom}(H, H)$, or find a counterexample.

Exercise 5.16 (**). Prove that the Calkin algebra does not contain non-trivial closed two-sided ideals.

Exercise 5.17. Let A be an invertible operator on a Hilbert space H. Prove that $A + B$ is invertible for any $B \in \operatorname{End}(H)$ such that $\|B\| < \|A^{-1}\|^{-1}$.

Exercise 5.18 (!). Let A be a Fredholm operator on a Hilbert space H, and $A_0: H/\ker A \longrightarrow \operatorname{im} A$ the corresponding invertible operator. Prove that $A + B$ is Fredholm for any $B \in \operatorname{End}(H)$ such that $\|B\| < \|A_0^{-1}\|^{-1}$.

Exercise 5.19. Let A be a Fredholm operator on a Hilbert space A, and R a finite rank operator. Prove that $A+R$ is Fredholm.

Exercise 5.20 (!). Let A be a Fredholm operator on a Hilbert space H, and K a compact operator. Prove that $A+K$ is Fredholm.

Hint. Use Exercises 5.18 and 5.19.

Exercise 5.21 (*). Let A be a Fredholm operator on a Hilbert space A. Prove that there exists a compact operator K such that $A+K$ is invertible, or find a counterexample.

Exercise 5.22 (!). Let A be an operator on a Hilbert space H. Prove that A is Fredholm if and only if its class in Calkin algebra is invertible in Calkin algebra.

Exercise 5.23 (**). Prove that the Calkin algebra C is equipped with a norm, that is, a function $\nu : C \longrightarrow \mathbb{R}^{\geqslant 0}$ such that $\nu(c) > 0$ for all non-zero c, $\nu(x+y) \leqslant \nu(x) + \nu(y)$ and $\nu(xy) \leqslant \nu(x)\nu(y)$.