Hodge theory 6: Connections

Rules: You may choose to solve only "hard" exercises (marked with !, * and **) or "ordinary" ones (marked with ! or unmarked), or both, if you want to have extra stuff to work. To have a perfect score, a student must obtain (in average) a score of 10 points per week.

If you have got credit for 2/3 of ordinary problems or 2/3 of "hard" problems, you receive $6t$ points, where t is a number depending on the date when it is done. Passing all "hard" or all "ordinary" problems brings you $10t$ points. Solving of "**" (extra hard) problems is not obligatory, but each such problem gives you a credit for 2 "*" or "!" problems in the "hard" set.

The first 3 weeks after giving a handout, $t = 1.5$, between 21 and 35 days, $t = 1$, and afterwards, $t = 0.7$.

The scores are not cumulative, only the best score for each handout counts.

6.1 Connections

Definition 6.1. Let B be a vector bundle on a smooth manifold M, and

$$\nabla : B \longrightarrow B \otimes \Lambda^1 M$$

a differential operator which satisfies

$$\nabla(fb) = b \otimes df + f\nabla b,$$

for any $f \in C^\infty(M)$ and any $b \in B$. Then ∇ is called **connection** on B. Given a vector field X, consider an operator $\nabla_X : B \longrightarrow B$ obtained by the convolution of $\nabla(b)$ with X. This operator is called **covariant derivative** along X.

Exercise 6.1. Prove that the covariant derivative ∇_X satisfies the Leibniz rule $\nabla_X(fb) = \langle df, X\rangle b + f\nabla_X b$. Here $\langle df, X = \operatorname{Lie}_X f = df \lrcorner X$ denotes the derivative of f along X.

Exercise 6.2. Let B be a vector bundle on M. Suppose that for any vector field $X \in TM$ we are provided with a covariant derivative operator $\nabla_X : B \longrightarrow B$ satisfying the Leibniz rule and $C^\infty M$-linear on X. Prove that ∇_X is obtained from a connection.

Exercise 6.3. Prove that the symbol $\operatorname{Symb}(\nabla) \in TM \otimes \operatorname{Hom}(B, \Lambda^1 M \otimes B)$ of a connection is given by an identity map $\mathsf{Id}_{TM} \otimes \mathsf{Id}_B : TM \otimes T^*M \otimes \operatorname{Hom}(B, B)$.

Exercise 6.4 (!). Let D be a first order differential operator on a bundle B with symbol $\operatorname{Symb}(D) \in TM \otimes \operatorname{Hom}(B, \Lambda^1 M \otimes B)$ given by an identity map $\mathsf{Id}_{TM} \otimes \mathsf{Id}_B : TM \otimes T^*M \otimes \operatorname{Hom}(B, B)$. Prove that it is a connection.

Exercise 6.5 (!). Let ∇ be a connection on B, B^* the dual bundle. Prove that there exists a unique operator $\nabla^* : B^* \longrightarrow B^* \otimes \Lambda^1 M$ such that

$$d\langle b, b'\rangle = \langle \nabla b, b'\rangle + \langle b, \nabla^* b'\rangle$$

for any $b \in B, b' \in B^*$. Prove that ∇^* is a connection on B^*.

Exercise 6.6. Let $B_1, ..., B_n$ be vector bundles with connections, denoted by ∇ (people often use the same letter ∇ to denote different connections if they are defined on different bundles). Consider the following differential operator

$$\nabla : B_1 \otimes ... \otimes B_n \longrightarrow B_1 \otimes ... \otimes B_n \otimes \Lambda^1 M,$$

$\nabla(b_1 \otimes b_2 \otimes ... \otimes b_n) = \nabla(b_1) \otimes b_2 \otimes ... \otimes b_n + b_1 \otimes \nabla(b_2) \otimes ... \otimes b_n + ... + b_1 \otimes b_2 \otimes ... \otimes \nabla(b_n)$. Prove that ∇ defines a connection on the vector bundle $B_1 \otimes ... \otimes B_n$.

Remark 6.1. Previous two exercises show that a connection on a bundle B defines a connection on any tensor power $B^{\otimes n} \otimes (B^*)^{\otimes m}$. This connection is almost always denoted by the same letter.

Exercise 6.7 (!). Let B be a vector bundle over a manifold admitting partition of unity. Prove that B admits a connection.

6.2 Holonomy

Definition 6.2. Let (B, ∇) be a bundle with connection. A tensor $\Psi \in B^{\otimes i} \otimes (B^*)^{\otimes j}$ is called **parallel** if $\nabla(\Psi) = 0$. In this case we also say that Ψ **is preserved** by ∇.

Exercise 6.8. Let g be a tensor on a bundle B over M. Construct a connection ∇ such that $\nabla(g) = 0$ if

a. (!) g is a non-degenerate bilinear symmetric form on B.

b. (*) g is a non-degenerate bilinear antisymmetric form on B.

c. (*) g is a bilinear symmetric form of constant rank on B.

Exercise 6.9. Let B be a trivial vector bundle with connection over $\mathbb{R}$. Prove that for each $x \in \mathbb{R}$ and each vector $b_x \in B\Big|_x$ there exists a unique section $b \in B$ such that $\nabla b = 0$, $b\Big|_x = b_x$.

Definition 6.3. Let $\gamma : [0,1] \longrightarrow M$ be a smooth path in M connecting x and y, and (B, ∇) a vector bundle with connection. Restricting (B, ∇) to $\gamma[0,1]$, we obtain a bundle with connection on an interval. Solve an equation $\nabla(b) = 0$ for $b \in B\Big|_{\gamma([0,1])}$ and initial condition $b\Big|_x = b_x$. This process is called **parallel transport** along the path via the connection. The vector $b_y := b\Big|_y$ is called **vector obtained by parallel transport of b_x along γ. Holonomy group** of γ is the group of endomorphisms of the fiber B_x obtained from parallel transports along all paths starting and ending in $x \in M$

Exercise 6.10. Let (B, ∇) be a vector bundle over a connected manifold M, and $x, y \in M$. Construct an isomorphism of the corresponding holonomy groups $\mathrm{Hol}_x(\nabla) \longrightarrow \mathrm{Hol}_y(\nabla)$.

Exercise 6.11. Find a bundle with connection over S^1 which has non-trivial holonomy.

6.3 Iterated connection

Definition 6.4. Let M be a smooth manifold. A connection on TM or on $\Lambda^1 M$ is called **connection on** M. This connection defines a connection on all tensor powers of TM and $\Lambda^1 M$. A tensor product of several copies of TM and $\Lambda^1 M$ is called **a tensor bundle** on M, and its section **a tensor**. Similarly, a section of a tensor product of several copies of B and B^* is called **a tensor over a bundle** B.

Definition 6.5. Let B be a vector bundle with connection ∇_0 over a manifold M, and ∇ a connection on $\Lambda^1 M$. Define a connection

$$\nabla_i:\ B\otimes \underbrace{\Lambda^1 M\otimes ...}_{i \text{ times}} \longrightarrow B\otimes \underbrace{\Lambda^1 M\otimes ...}_{i+1 \text{ times}} \tag{6.1}$$

using the Leibniz formula

$$\nabla_i(b\otimes\xi_1\otimes ...\otimes \xi_i) = \nabla_{i-1}(b\otimes\xi_1\otimes ...\otimes\xi_{i-1})\otimes\xi_i + b\otimes\xi_1\otimes ...\otimes\xi_{i-1}\otimes\nabla\xi_i.$$

Denote by

$$\nabla^i:\ B\longrightarrow B\otimes \underbrace{\Lambda^1 M\otimes ...}_{i \text{ times}}$$

the composition $\nabla_0\circ\nabla_1\circ ...\circ\nabla_i$. This operator is called **an i-th power of the connection** ∇.

Exercise 6.12. a. Prove that the symbol of ∇^2, considered as an element of

$$\mathsf{Sym}^2\, TM\otimes \operatorname{Hom}(B, B\otimes\Lambda^1 M\otimes\Lambda^1 M)$$

is symmetric under the permutation of the tensor multipliers $\Lambda^1 M\otimes\Lambda^1 M$.

b. (*) Let S be the symbol of ∇^i,

$$S\in \mathsf{Sym}^i\, TM\otimes \operatorname{Hom}\left(B, B\otimes\underbrace{\Lambda^1 M\otimes ...}_{i}\right)$$

Prove that S is symmetric under the permutations of the tensor multipliers $\Lambda^1 M\otimes\Lambda^1 M\otimes ...\otimes\Lambda^1 M$.

Exercise 6.13. Let $D\in \operatorname{Diff}^s(B,C)$ be a differential operator on vector bundles B, C.

a. (!) Prove that there exists a C^∞-linear map

$$\Psi:\ B\otimes\bigoplus_{i=0}^{s}(\Lambda^1 M)^{\otimes i}\longrightarrow C$$

such that $D(b) = \Psi\left(\bigoplus_{i=0}^s \nabla^i b\right)$.

b. (**) Prove that there exists a C^∞-linear map and a connection ∇ such that $\Phi:\ B\otimes(\Lambda^1 M)^{\otimes s}\longrightarrow C$ such that $D(b)=\Phi(\nabla^s b)$.

6.4 Adjoint operators

Exercise 6.14 (!). Let $D:\ \ B\longrightarrow C$ be a first order differential operator on a vector bundle with connection ∇. Prove that D is obtained as a linear combination of $b\longrightarrow A(\nabla_{X_i})$, where X_i are vector fields and $A:\ B\longrightarrow C$ is a linear map.

Definition 6.6. Let Vol be a volume form on a manifold M. **Divergence** of a vector field X is the form $\operatorname{Lie}_X \operatorname{Vol} = d(\operatorname{Vol} \lrcorner X)$.

Exercise 6.15 (!). Let Vol be a volume form on a manifold M, and $\mathcal{D}\subset TM$ the sheaf of vector fields with zero divergence. Prove that $\mathcal{D}$ generates TM over $C^\infty M$.

Exercise 6.16. Let B be a vector bundle over a manifold M, Vol a volume form and ∇ a connection on B. Consider the pairing between sections of B and the dual bundle B^*, with

$$\langle b,\lambda\rangle=\int_M (b,\lambda)\operatorname{Vol} \tag{6.2}$$

for each $b\in B,\lambda\in B^*$.

a. Let $X\in TM$ a vector field with zero divergence. Prove that the operators $\nabla_X:\ \ B\longrightarrow B$ and $\nabla^*_X:\ \ B^*\longrightarrow B^*$ are adjoint with respect to the pairing (6.2).

b. Let X be an arbitrary vector field, and $\nabla_X:\ \ B\longrightarrow B$ the corresponding covariant derivative. Consider the adjoint $(\nabla_X)^*:\ B^*\longrightarrow B^*$ associated with the pairing (6.2). Prove that

$$\int_M (b,\nabla^*_X b')\operatorname{Vol}+\int_M(\nabla_X b,b')\operatorname{Vol}=-\int_M (b,b')\operatorname{Lie}_X\operatorname{Vol}$$

c. (!) Prove that $(\nabla_X)^*$ is a differential operator.

Exercise 6.17. Prove that an adjoint map of a first order differential operator on vector bundles $D:\ B\longrightarrow B_1$ is a first order differential operator from B_1^* to B^*.

Hint. Use Exercise 6.14.