Hodge theory 7: Sobolev L_p^2 norms

Rules: You may choose to solve only "hard" exercises (marked with !, * and **) or "ordinary" ones (marked with ! or unmarked), or both, if you want to have extra stuff to work. To have a perfect score, a student must obtain (in average) a score of 10 points per week.

If you have got credit for 2/3 of ordinary problems or 2/3 of "hard" problems, you receive $6t$ points, where t is a number depending on the date when it is done. Passing all "hard" or all "ordinary" problems brings you $10t$ points. Solving of "**" (extra hard) problems is not obligatory, but each such problem gives you a credit for 2 "*" or "!" problems in the "hard" set.

The first 3 weeks after giving a handout, $t = 1.5$, between 21 and 35 days, $t = 1$, and afterwards, $t = 0.7$. The scores are not cumulative, only the best score for each handout counts.

7.1 Sobolev L_p^2-spaces

Definition 7.1. Hilbert basis in a Hilbert space H is a set of linearly independent, orthogonal vectors generating a space H_0 which is dense in H.

Exercise 7.1. Consider the space of complex-valued functons on an n-dimensional compact torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ with the metric given by $|f|^2 = \int_{T^n} |f|^2 \operatorname{Vol}$, where Vol is the standard volume form on a torus, induced from $\mathbb{R}^n$. Denote by $L^2(T^n)$ the Hilbert space obtained as a completion of $C^\infty(T^n)$ with this metric. Prove that the following functions constitute an orthonormal basis in $L^2(T^n)$

$$\exp\left(2\pi\sqrt{-1}\sum_{i=1}^n k_i t_i\right) \tag{7.1}$$

where $k_1, ..., k_n$ run through the set $\mathbb{Z}^n$ of all integer n-tuples.

Definition 7.2. The functions (7.1) are called **the Fourier monomials** on a torus.

Definition 7.3. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth function with compact support. For any differential monomial $P_\alpha = \frac{\partial^{k_1}}{\partial x_1^{k_1}} \frac{\partial^{k_2}}{\partial x_2^{k_2}} \cdots \frac{\partial^{k_n}}{\partial x_1^{k_n}}$ consider the corresponding partial derivative $P_\alpha(f) = \frac{\partial^{k_1}}{\partial x_1^{k_1}} \frac{\partial^{k_2}}{\partial x_2^{k_2}} \cdots \frac{\partial^{k_n}}{\partial x_1^{k_n}} f$. **The L_p^2 Sobolev norm** $|f|_p$ is defined as follows:

$$|f|_p^2 = \sum_{\deg P_\alpha \leqslant p} \int |P_\alpha(f)|^2 \operatorname{Vol}$$

where the sum runs through all differential monomials of degree $\leqslant p$, and Vol is the standard volume form. This is a positive definite quadratic form on the space $C_c^\infty(\mathbb{R}^n)$ of functions with compact support, and its square root gives the norm. The Sobolev L_p^2 norm on sections of trivial bundle is defined the same way. Also, the same formula can be used to define the L_p^2-norm on functions on a torus.

Exercise 7.2. Consider the Fourier series for the function f:

$$f = \sum_{k_1,\dots k_n \in \mathbb{Z}^n} \tau_{k_1,\dots k_n} e^{2\pi\sqrt{-1}\sum_{i=1}^n k_i t_i}$$

a. Prove that the L^2_p-norm can be written as

$$|f|^2_s = \sum_{k_1,...k_n\in\mathbb{Z}^n}\left(|\tau_{k_1,...k_n}|^2\sum_{i=1}^n \Psi(k_1,...,k_n)\right),$$

where $\Psi(k_1,...,k_n) = \sum_\alpha (2\pi)^{2d}|P_\alpha(k_1,...,k_n)|^2$, where P_α runs through all monomials of degree $d \leqslant p$.

b. (!) Prove that L^2_p-norm is equivalent to the norm

$$|f|^2_{s,\bullet} = \sum_{k_1,...k_n\in\mathbb{Z}^n}\left(|\tau_{k_1,...k_n}|^2\sum_{i=1}^n 1+k_i^{2p}\right)$$

Exercise 7.3 (!). (Rellich lemma)
Prove that the identity map $L^2_s(\mathbb{R}^n) \longrightarrow L^2_{s-1}(\mathbb{R}^n)$ is compact on the Hilbert space generated by functions with support in an open ball of radius R, for any given $R > 0$.

Hint. Use the previous exercise.

Exercise 7.4 (*). Is the map $L^2_s(\mathbb{R}^n) \longrightarrow L^2_{s-1}(\mathbb{R}^n)$ compact on the Hilbert space generated by all functions with compact support?

Exercise 7.5. Consider the Fourier series of one variable

$$\sum_{k\in\mathbb{Z}} \tau_k e^{2\pi\sqrt{-1}kt} \tag{7.2}$$

Supppose that $\sum_{k\in\mathbb{Z}} k^{2+2l}|\tau_k|^2$ converges. Prove that (7.2) converges to a function of class C^l.

Exercise 7.6. Prove an inequality

$$\left(\sum_{k_1,...k_n\in\mathbb{Z}^n\backslash 0}\frac{|\gamma_{k_1,...k_n}|}{\sum\left|k_i^{n+1}\right|}\right)^2 \leqslant \left(\sum_{k_1,...k_n\in\mathbb{Z}^n}|\gamma_{k_1,...k_n}|^2\right)\left(\sum_{k_1,...k_n\in\mathbb{Z}^n}\frac{1}{\sum|k_i|^{n+2}})\right)$$

Hint. Use the Cauchy-Schwarz inequality.

Exercise 7.7. Consider the Fourier series

$$\sum_{k_1,...k_n\in\mathbb{Z}^n}\tau_{k_1,...k_n}e^{2\pi\sqrt{-1}\sum_{i=1}^n k_it_i}. \tag{7.3}$$

a. Suppose that the series

$$\sum_{k_1,...k_n\in\mathbb{Z}^n}\left(|\tau_{k_1,...k_n}|^2\sum_{i=1^n}k_i^{n+1}\right)$$

converges. Prove that (7.3) converges to a continuous function.

b. (!) Suppose that the series

$$\sum_{k_1,\dots k_n\in\mathbb{Z}^n} |\tau_{k_1,\dots k_n}|^2 \sum_{i=1}^n |k_i|^{n+2+2l}$$

converges. Prove that (7.3) converges to a function from $C^l(T^n)$ (l times differentiable).

Hint. Use the Cauchy-Schwarz inequality.

Exercise 7.8 (!). (Sobolev's lemma)
Let $\{f_i\}$ be a sequence of smooth functions with support in a ball $B\subset\mathbb{R}^n$ converging in the Sobolev L_s^2-norm, and $s > l+\frac{n}{2}$. Prove that $\{f_i\}$ converges in $L_s^2(\mathbb{R}^n)$-topology to a function from $C^l(\mathbb{R}^n)$.

Hint. Use the previous exercise.

Exercise 7.9 (*). Is this statement true for any $\{f_i\}$ in $C_c^\infty(\mathbb{R}^n)$, without assuming that f_i are supported in a ball B?

Exercise 7.10 (!). Let $\sum\psi_i = 1$ be a partition of unity on $\mathbb{R}^n$, with $\operatorname{Supp}(\psi_i)\subset U_i$, where U_i is a bounded set. Let $C_c^\infty(\mathbb{R}^n)$ denote the space of functions with compact support, and $C_{U_i}^\infty(\mathbb{R}^n)$ the space of functions with support in U_i. Consider the map $C_c^\infty(\mathbb{R}^n)\longrightarrow\bigoplus_i C_{U_i}^\infty(\mathbb{R}^n)$ mapping f to $\bigoplus_i\psi_i f$. Prove that (for an appropriate partition of unity) it can be extended to the L_p^2-completions of the relevant spaces $\Psi: L_p^2(\mathbb{R}^n)\longrightarrow\bigoplus_i L_p^2(\mathbb{R}^n)_{U_i}$, where $L_p^2(\mathbb{R}^n)_{U_i}$ denotes the completion of $C_{U_i}^\infty(\mathbb{R}^n)$. Prove that Ψ has closed image in $\bigoplus_i L_p^2(\mathbb{R}^n)_{U_i}$.

7.2 Sobolev L_p^2-norms on a manifold

Definition 7.4. Let g be a Riemannian metric on a manifold M, and F a vector bundle on M with connection and a Euclidean or Hermitian metric. Define **the Sobolev L_s^2-norm associated with the metric and the connection** by the formula

$$|f|_s^2 = \sum_{i=0}^s \int |\nabla^i f|^2 \operatorname{Vol} \tag{7.4}$$

where $\nabla^i: F\longrightarrow F\otimes\underbrace{\Lambda^1 M\otimes\dots}_{i\text{ times}}$ is the i-th power of connection defined in Handout 6, and $|\cdot|^2$ the standard Euclidean (or Hermitian) metric.

Exercise 7.11. Suppose that M is a torus, B trivial, the metrics on B and TM are standard, and ∇, ∇_0 the standard connections written in stanard coordinates as $\nabla\left(\sum f_i\xi_i\right) = \sum df_i\xi_i$. Prove that in this case the L_s^2-norm defined in (7.4) is equivalent to the one defined in Subsection 7.1.

Exercise 7.12 (!). Let ν and ν' be Sobolev L^2_s-norms on B associated with some connections and metrics (generally speaking, different). Prove that the corresponding topologies are equivalent. Assume that the base manifold is compact.

Hint. Express arbitrary differential operator through iterated connections like in Handout 6.

Exercise 7.13 (!). Let $\sum \psi_i = 1$ be a partition of unity on a compact manifold M, with $\operatorname{Supp}(\psi_i) \subset U_i$, where U_i is a bounded set. Let $C^\infty_{U_i}(M)$ the space of functions with support in U_i. Consider the map $C^\infty(M) \longrightarrow \bigoplus_i C^\infty_{U_i}(M)$ mapping f to $\bigoplus_i \psi_i f$. Prove that it can be extended to the L^2_p-completions of the relevant spaces, $\Psi : L^2_p(M) \longrightarrow \bigoplus_i L^2_p(M)_{U_i}$, where $L^2_p(M)_{U_i}$ denotes the completion of $C^\infty_{U_i}(M)$. Prove that Ψ has closed image in $\bigoplus_i L^2_p(M)_{U_i}$.

Exercise 7.14 (!). (Rellich lemma)
Let M be a compact manifold, F a vector bundle on M, and $L^2_s(F) \longrightarrow L^2_{s-i}(F)$ the identity map. Prove that this is a compact operator for all $i > 0$.

Hint. Use the Rellich lemma for torus (Exercise 7.3) and a partition of unity.

Exercise 7.15 (*). Is this statement true for the L^2_s-space obtained as a completion of sections with compact support on a non-compact manifold?

Exercise 7.16 (!). Let $\{f_i\}$ be a sequence of smooth sections of a bundle B on a compact manifold M converging in the Sobolev L^2_s-norm, and $s > l + \frac{n}{2}$. Prove that $\{f_i\}$ converges in L^2_s-topology to a section which is l times differentiable.

Hint. Use the Sobolev lemma for a torus (Exercise 7.8) and a partition of unity.

Exercise 7.17 (*). Prove the Sobolev lemma for the space of functions with compact support on a non-compact manifold.

Exercise 7.18 (!). Using the injective maps $L^2_s(B) \longrightarrow L^2_{s-i}(B)$, we can consider all these spaces as subspaces in $L^2(B)$. Prove that $\bigcap_s L^2_s(B)$ is the space of smooth sections of B.

Exercise 7.19. Let $D_1, D_2 : F \longrightarrow F$ be differential operators of order i, j on a vector bundle F on a compact manifold. Prove that the commutator $[D_1, D_2] : L^2_s(F) \longrightarrow L^2_{s-i-j}(F)$ is compact, or find a counterexample.