# Hodge theory 8: Cartan's formula and Poincaré lemma

**Rules:** You may choose to solve only "hard" exercises (marked with !, * and **) or "ordinary" ones (marked with ! or unmarked), or both, if you want to have extra stuff to work. To have a perfect score, a student must obtain (in average) a score of 10 points per week.

If you have got credit for 2/3 of ordinary problems or 2/3 of "hard" problems, you receive $6t$ points, where $t$ is a number depending on the date when it is done. Passing all "hard" or all "ordinary" problems brings you $10t$ points. Solving of "**" (extra hard) problems is not obligatory, but each such problem gives you a credit for 2 "*" or "!" problems in the "hard" set.

The first 3 weeks after giving a handout, $t = 1.5$, between 21 and 35 days, $t = 1$, and afterwards, $t = 0.7$. The scores are not cumulative, only the best score for each handout counts.

## 8.1 Lie derivative

**Definition 8.1.** An associative algebra $A^* = \oplus_{i\in\mathbb{Z}} A^i$ is called **a graded algebra** if for all $a \in A^i$, $b \in A^j$, the product $ab$ lies in $A^{i+j}$.

**Definition 8.2.** Let $A^* = \oplus_{i\in\mathbb{Z}} A^i$ be a graded algebra over a field. It is called **graded commutative**, or **supercommutative**, if $ab = (-1)^{ij} ba$ for all $a \in A^i, b \in A^j$.

**Remark 8.1.** Grassmann algebra $\Lambda^* V$ is clearly supercommutative.

**Definition 8.3.** Let $A^*$ be a graded commutative algebra, and $D: A^* \longrightarrow A^{*+i}$ a map which shifts grading by $i$. It is called a **graded derivation** if $D(ab) = D(a)b + (-1)^{ij} a D(b)$, for each $a \in A^j$.

**Remark 8.2.** If $i$ is even, graded derivation is just a derivation. If it is odd, it is called **odd derivation**.

**Remark 8.3.** De Rham differential is an odd derivation, by definition.

**Definition 8.4.** Let $M$ be a smooth manifold, and $X \in TM$ a vector field. Consider an operation of **convolution with a vector field** $i_X: \Lambda^i M \longrightarrow \Lambda^{i-1} M$, mapping an $i$-form $\alpha$ to an $(i-1)$-form $v_1, ..., v_{i-1} \longrightarrow \alpha(X, v_1, ..., v_{i-1})$

**Exercise 8.1.** Prove that $i_X$ is an odd derivation.

**Exercise 8.2 (*).** Let $D: A^* \longrightarrow A^{*+i}$ be a linear map such that for all $x \in A$ there exists $N$ such that $D^N(x) = 0$. Prove that $e^D := 1 + D + \frac{D^2}{2} + ... + \frac{D^i}{i!} + ...$ is an automorphism of $A^*$ if and only if $D$ is a derivation.

**Definition 8.5.** Let $A^*$ be a graded vector space, and $E: A^* \longrightarrow A^{*+i}$, $F: A^* \longrightarrow A^{*+j}$ operators shifting the grading by $i, j$. Define **the supercommutator** by the formula

$$\{E, F\} := EF - (-1)^{ij} FE.$$

**Remark 8.4.** An endomorphism which shifts a grading by $i$ is called **even** if $i$ is even, and **odd** otherwise.

**Remark 8.5.** Supercommutator satisfies **graded Jacobi identity,**

$$\{E, \{F, G\}\} = \{\{E, F\}, G\} + (-1)^{\tilde{E}\tilde{F}} \{F, \{E, G\}\}$$

where $\tilde{E}$ and $\tilde{F}$ are 0 if $E, F$ are even, and 1 otherwise.

**Remark 8.6.** There is a simple mnemonic rule which allows one to remember a superidentity, if you know the commutative analogue. Each time when in commutative case two letters $E$, $F$ are exchanged, in supercommutative case one needs to multiply by $(-1)^{\tilde{E}\tilde{F}}$.

**Exercise 8.3.** Let $A^*$ be a graded commutative algebra and $a \in A$. Denote by $L_a : A \longrightarrow A$ the operation of multiplication by $a$: $L_a(b) = ab$. Prove that a map $D : A^* \longrightarrow A^*$ is a superderivation if and only if $D(1) = 0$ and for each $a \in A^i$, the supercommutator $\{D, L_a\}$ is equal to $L_b$ for some $b \in A^*$.

**Exercise 8.4 (!).** Prove that a supercommutator of superderivations is again a superderivation.

**Hint.** Use the Jacobi identity and apply the previous exercise.

**Definition 8.6.** Let $B$ be a smooth manifold, and $v \in TM$ a vector field. An endomorphism $\operatorname{Lie}_v : \Lambda^* M \longrightarrow \Lambda^* M$, preserving the grading is called **a Lie derivative along** $v$ if it satisfies the following conditions.

**(i)** On functions $\operatorname{Lie}_v$ is equal to a derivative along $v$.

**(ii)** $[\operatorname{Lie}_v, d] = 0$

**(iii)** $\operatorname{Lie}_v$ is a derivation on the de Rham algebra.

**Exercise 8.5.** Let $\nu_1, \nu_2 : \Lambda^*(M) \longrightarrow \Lambda^*(M)$ be derivations of the de Rham algebra. Suppose that $\nu_1$ is equal to $\nu_2$ on $C^\infty M = \Lambda^0(M)$ and on $d(C^\infty M)$. Prove that $\nu_1 = \nu_2$.

**Hint.** $\Lambda^*(M)$ is generated (multiplicatively) by $C^\infty M = \Lambda^0(M)$ and $d(C^\infty M)$.

**Exercise 8.6.** Prove that the Lie derivative is uniquely determined by the properties (i)-(iii).

**Hint.** Use the previous exercise.

**Exercise 8.7.** Prove that $\{d, \{d, E\}\} = 0$, for each $E \in \operatorname{End}(\Lambda^* M)$.

**Hint.** Use the graded Jacobi identity.

**Exercise 8.8.** Prove that $\{d, i_v\}$ commutes with $d$, where $i_v : \Lambda^* M \longrightarrow \Lambda^{*-1} M$ is a convolution with $v$.

**Hint.** Use the previous exercise.

**Exercise 8.9 (!).** (Cartan formula) Prove that $\{d, i_v\}$ is a Lie derivative along $v$.

**Exercise 8.10 (*).** Let $\tau : \Lambda^*(M) \longrightarrow \Lambda^{*-1}(M)$ be a derivation shifting grading by $-1$. Prove that there exists a vector field $v \in TM$ such that $\tau = i_v$, or find a counterexample.

## 8.2 Poincaré lemma

**Exercise 8.11.** Let $t$ be the coordinate function on a real line, $f(t) \in C^\infty \mathbb{R}$ a smooth function, and $v := t\frac{d}{dt}$ a vector field. Define

$$R(f)(t) := \int_0^1 \frac{f(\lambda t)}{\lambda} d\lambda$$

Prove that this integral converges whenever $f(0) = 0$, and satisfies $\operatorname{Lie}_v R(f) = f$ in this case.

**Exercise 8.12.** Let $t_1, ..., t_n$ be coordinate functions in $\mathbb{R}^n$, and $\vec{r} := \sum_i t_i \frac{d}{dt_i}$ a radial vector field. Consider a function $f \in C^\infty \mathbb{R}^n$ satisfying $f(0) = 0$, and let $x = (x_1, ..., x_n)$ be any point in $\mathbb{R}^n$. Prove that an integral

$$R(f)(x) := \int_0^1 \frac{f(\lambda x)}{\lambda} d\lambda$$

converges, and satisfies $\operatorname{Lie}_{\vec{r}} R(f) = f$.

**Hint.** Use the previous exercise.

**Definition 8.7.** An open subset $U \subset \mathbb{R}^n$ is called **starlike** if for any $x \in U$ the interval $[0, x]$ belongs to $U$.

**Exercise 8.13 (!).** Let $U$ be a starlike subset in $\mathbb{R}^n$, and $i > 0$. Construct an operator

$$R : \Lambda^i U \longrightarrow \Lambda^i U$$

which satisfies $\operatorname{Lie}_{\vec{r}} R\alpha = R \operatorname{Lie}_{\vec{r}} \alpha = \alpha$ for each $\alpha \in \Lambda^i U$.

**Hint.** Define the integral $R(\alpha)$ as in the previous exercise, and check that it converges. Prove that $\operatorname{Lie}_{\vec{r}} R(\alpha) = \alpha$.

**Exercise 8.14.** Prove that any form $\alpha \in \Lambda^i U$ on a starlike set $U$ satisfying $\operatorname{Lie}_{\vec{r}} \alpha = 0$ vanishes if $i > 0$.

**Hint.** Use the previous exercise.

**Exercise 8.15 (!).** Prove that $\{R, d\} = 0$

**Hint.** Check that

$$\{R, d\} \operatorname{Lie}_{\vec{r}} \alpha = Rd \operatorname{Lie}_{\vec{r}} \alpha + dR \operatorname{Lie}_{\vec{r}} \alpha = -R \operatorname{Lie}_{\vec{r}} d\alpha + d\alpha = 0.$$

For any $\beta \in \ker\{R, d\} \cap \Lambda^i M$, satisfying $i > 0$ or $\beta(0) = 0$ for $i = 0$, solve an equation $\operatorname{Lie}_{\vec{r}} \alpha = \beta$.

**Exercise 8.16.** Prove that $\{d, i_{\vec{r}}\} R(\alpha) = \alpha$, for any $i$-form $\alpha$ on a starlike set, $i > 0$.

**Definition 8.8.** Let $d$ be de Rham differential. A form in $\ker d$ is called **closed**, a form in $\operatorname{im} d$ is called **exact**. Since $d^2 = 0$, any exact form is closed. **The group of $i$-th de Rham cohomology of** $M$, denoted $H^i(M)$, is a quotient of a space of closed $i$-forms by exact: $H^*(M) = \frac{\ker d}{\operatorname{im} d}$.

**Exercise 8.17 (!).** Let $\alpha \in \Lambda^i U$ be a closed $i$-form on a starlike set $U$, with $i > 0$. Prove that $\alpha = d i_{\vec{r}} R(\alpha)$.

**Hint.** Use the previous exercise.

**Exercise 8.18 (!).** (Poincaré lemma) Let $U$ be a starlike set. Prove that $H^i(U) = 0$ for each $i > 0$, and $H^0(M) = \mathbb{R}$.

**Exercise 8.19.** Let $\theta$ be a closed odd form, and $d_\theta(x) = dx + \theta \wedge x$ the corresponding operator on $\Lambda^* M$. Its **cohomology** are defined as $H^*(\Lambda^*(M), d_\theta) := \frac{\ker d_\theta}{\operatorname{im} d_\theta}$

a. Show that $d_\theta^2 = 0$.

b. (*) Let $\theta$ be an exact 1-form. Prove that $H^i(\Lambda^*(M), d_\theta)$ are isomorphic to $H^i(M)$.

c. (*) Let $\theta$ be a closed 1-form. Prove that $H^i(\Lambda^*(M), d_\theta)$ are isomorphic to $H^i(M)$, or find a counterexample.

d. (*) Let $\theta$ be a closed 3-form. Prove that $H^*(\Lambda^*(M), d_\theta)$ are isomorphic to $H^*(M)$, or find a counterexample.