Hodge theory 9: Elliptic operators

Rules: You may choose to solve only "hard" exercises (marked with !, * and **) or "ordinary" ones (marked with ! or unmarked), or both, if you want to have extra stuff to work. To have a perfect score, a student must obtain (in average) a score of 10 points per week.

If you have got credit for 2/3 of ordinary problems or 2/3 of "hard" problems, you receive $6t$ points, where t is a number depending on the date when it is done. Passing all "hard" or all "ordinary" problems brings you $10t$ points. Solving of "**" (extra hard) problems is not obligatory, but each such problem gives you a credit for 2 "*" or "!" problems in the "hard" set.

The first 3 weeks after giving a handout, $t = 1.5$, between 21 and 35 days, $t = 1$, and afterwards, $t = 0.7$. The scores are not cumulative, only the best score for each handout counts.

9.1 Hodge $*$ operator

Using the Riemannian metric on a manifold M, we can construct a natural metric on all tensor powers of the tangent and cotangent bundle, in particular, on $\Lambda^*(M)$. We normalize the metric on $\Lambda^k(M)$ in such a way that the basis

$$\xi_{i_1} \wedge \xi_{i_2} \wedge ... \wedge \xi_{i_k}, \quad i_1 < i_2 < ... < i_k$$

is orthonormal, for an orthonormal frame $\xi_1, ..., \xi_n \in T^*M$.

Definition 9.1. Let M be an oriented Riemannian manifold, and $\xi_1, ..., \xi_n$ an oriented orthonormal frame in T^*M. **The Riemannian volume form** Vol is $\xi_1 \wedge \xi_2 \wedge ... \wedge \xi_n$. Further on, all Riemannian manifolds are assumed oriented and equipped with Riemannian volume.

Exercise 9.1. Let $\eta \in \Lambda^k M$ be a differential form, and $*\eta \in \Lambda^{n-k}M$ a form which satisfies $(\eta, \xi) \operatorname{Vol} M = *\eta \wedge \xi$ for any $\xi \in \Lambda^k M$.

a. Prove that $*\eta$ is uniquely determined by this relation.

b. Prove that $*\eta$ exists for any η.

c. Prove that in the basis $\xi_{i_1} \wedge \xi_{i_2} \wedge ... \wedge \xi_{i_k}$ defined above, the operator $*$ is written as follows:

$$*\xi_{i_1} \wedge \xi_{i_2} \wedge ... \wedge \xi_{i_k} = \sigma \xi_{j_1} \wedge \xi_{j_2} \wedge ... \wedge \xi_{j_{n-k}}$$

where $\{j_1, j_2, ... j_{n-k}\} := \{1, ..., n\} \backslash \{i_1, i_2, ..., i_k\}$, ordered by $j_1 < j_2 < ...$, and $\sigma = \pm 1$ is the signature of the permutation $(j_1, j_2, ... j_{n-k}, i_1, i_2, ..., i_k)$.

Exercise 9.2. Let ξ be a k-form, and η an $(n-k)$-form. Prove that

a. $(*\eta, \xi) = (-1)^{k(n-k)}(\eta, *\xi)$

b. $*(*\eta) = (-1)^{k(n-k)}\eta$.

Exercise 9.3 (!). Find the eigenvalues of $*$ on $\Lambda^{\frac{n}{2}}M$ for even-dimensional M, and dimension of the eigenspaces.

Exercise 9.4 (*). Let M be a pseudo-Riemannian manifold with the metric of signature $(n-s,s)$. Prove that $*(*\eta) = (-1)^{k(n-k)+s}\eta$.

Exercise 9.5 (!). Define $d^*: \Lambda^k(M) \longrightarrow \Lambda^{k-1}(M)$ using the formula $d^* := (-1)^{(n+1)(k+1)} * d *$. Prove that for any form ξ with compact support, one has

$$\int_M (d^*\eta, \xi) \operatorname{Vol} M = \int_M (\eta, d\xi) \operatorname{Vol} M.$$

Hint. Use the Stokes' formula:

$$\begin{aligned}\int_M (d^*\eta, \xi) \operatorname{Vol} M = \int_M (d^*\eta, \xi) \operatorname{Vol} M = -\int_M d * \eta \wedge \xi \\ = \int_M *\eta \wedge d\xi = \int_M (\eta, d\xi) \operatorname{Vol} M\end{aligned}$$

Remark 9.1. This implies that d and d^* are adjoint.

9.2 Laplace operator on differential forms

Definition 9.2. Define the Laplace operator Δ on differential forms using $\Delta\eta := dd^*\eta + d^*d\eta$.

Exercise 9.6. Let

$$\Lambda^k M \otimes \Lambda^1 M \xrightarrow{\ i\ } \Lambda^{k-1} M$$

be the "interior multiplication" map taking a tensor $\eta \otimes \theta \in \Lambda^k M \otimes \Lambda^1$ to $\eta \lrcorner \theta^\sharp$, where $\theta^\sharp$ is a vector field dual to θ. Prove that

$$\iota(\eta \otimes \theta) = (-1)^{(n+1)(k+1)} * (*\eta \wedge \theta).$$

Exercise 9.7. Consider the exterior multiplication map

$$\Lambda^k M \otimes \Lambda^1 M \xrightarrow{\ e\ } \Lambda^{k+1} M.$$

Let ∇ be a Levi-Civita connection on a Riemannian manifold M.

a. Prove that $d\eta = e(\nabla\eta)$.

b. Prove that $d^*\eta = \iota(\nabla\eta)$.

Hint. Prove that $d\eta = e(\nabla\eta)$ is equivalent to ∇ being torsion-free.

9.3 Elliptic operators

Definition 9.3. Let B be a vector bundle on M, and $\operatorname{Tot}(B)$ its total space. **Zero section** is the set of all zero vectors in $\operatorname{Tot} B$.

Definition 9.4. Let F, G be vector bundles on M. Consider the space $S^i(F,G) := \operatorname{Diff}^i(F,G)/\operatorname{Diff}^{i-1}(F,G) = \mathsf{Sym}^i(TM) \otimes \operatorname{Hom}(F,G)$. **Symbol** of a differential operator is its class $\sigma(D) \in S^i(F,G)$. The operator D is called **elliptic** if $\mathsf{rk}\, F = \mathsf{rk}\, G$, and for any non-zero point $\theta \in \operatorname{Tot}(T^*_x M)$ the symbol $\sigma(D)$ evaluated at $\underbrace{\theta \otimes ... \otimes \theta}_{i \text{ times}}$ is a non-degenerate as an element of $\operatorname{Hom}(F,G)\big|_x$.

Exercise 9.8. Let $D_1 : F \longrightarrow G$, $D_2 : G \longrightarrow H$ be differential operators, with F, G, H vector bundles of the same rank.

a. Prove that the composition $D_1 \circ D_2$ is elliptic if D_2, D_2 are elliptic.

b. Prove that D_1 and D_2 are elliptic if $D_1 \circ D_2$ is elliptic.

Exercise 9.9. Let $\Delta : C^\infty M \longrightarrow C^\infty M$ be the Laplace operator on a Riemannian manifold. Prove that it is elliptic.

Exercise 9.10 (!). Let $D : F \longrightarrow G$ be an elliptic operator, and D^* is Hermitian adjoint. Prove that D^* is also elliptic.

Exercise 9.11 (!). Let $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ be the connection, and $\nabla^* : B \otimes \Lambda^1 M \longrightarrow B$ its Hermitian adjoint. Prove that $\nabla^* \nabla$ is elliptic, and its symbol is equal to $-g \otimes \mathsf{Id}_B$, where $g \in \mathsf{Sym}^2 TM$ is the metric tensor.

Exercise 9.12 (!). Consider the operator $d + d^* : \Lambda^* M \longrightarrow \Lambda^* M$. Prove that it is elliptic.

9.4 Elliptic operators of second order

Exercise 9.13. Let $D : C^\infty M \longrightarrow C^\infty M$ be an elliptic operator of second order. Prove that $\sigma(D) \in \mathsf{Sym}^2(TM)$ is a positive definite or negative definite bilinear symmetric form.

Remark 9.2. From now till the end of this handout, $D : C^\infty M \longrightarrow C^\infty M$ is a differential operator of second order, the symbol of D is positive definite, and $D(1) = 0$.

Exercise 9.14. Let $f \in C^\infty M$ be a function which has a local maximum in z. Prove that $D(f)(z) \leqslant 0$.

Exercise 9.15 (!). Let $f \in C^\infty M$, and $D(f) > 0$. Prove that f cannot have a local maximum anywhere on M.

Exercise 9.16. Let

$$D(f) = \sum_{i,j} A^{ij} \frac{d^2 f}{dx_i dx_j} + \sum_i B^i \frac{df}{dx_i}, \tag{9.1}$$

be an elliptic operator, $D(f) \geqslant 0$, and $\lambda > 0$ a number which satisfies $\lambda A^{1,1} > -B^1$. Let $\phi_\varepsilon := \varepsilon e^{\lambda x_1}$. Prove that $D(f + \phi_\varepsilon) > 0$ for any $\varepsilon > 0$.

Exercise 9.17. Let $D: C^\infty \mathbb{R}^n \longrightarrow C^\infty \mathbb{R}^n$ be an elliptic operator defined as in (9.1), $\Omega \subset \mathbb{R}^n$ an open subset with compact closure, and f a function which reaches its maximum in Ω.

a. Prove that one may chose $\lambda > 0$ in such a way that $\lambda A^{1,1} > -B^1$ on Ω.

b. Let $\delta := \sup_\Omega f - \sup_{\partial\Omega} f > 0$. Chose ε in such a way that $\sup_\Omega \phi_\varepsilon < \frac{\delta}{2}$. Prove that $f + \phi_\varepsilon$ reaches its maximum inside Ω.

c. Prove that $D(f + \phi_\varepsilon) > D(f)$.

Exercise 9.18 (!). (weak maximum principle for elliptic operators).
Let $D: C^\infty M \longrightarrow C^\infty M$ be an elliptic operator of second order, with $D(1) = 0$ and positive definite symbol, and $f \in C^\infty M$ a function which satisfies $D(f) \geqslant 0$. Prove that f cannot have local maxima.

Hint. Use the previous exercise

Exercise 9.19 (*). (Strong Maximum Principle)
In the assumptions of the previous exercise, prove that $f(m) < \sup_M f$ for any $m \in M$.