Hodge theory 10: Green operators

Rules: You may choose to solve only "hard" exercises (marked with !, * and **) or "ordinary" ones (marked with ! or unmarked), or both, if you want to have extra stuff to work. To have a perfect score, a student must obtain (in average) a score of 10 points per week.

If you have got credit for 2/3 of ordinary problems or 2/3 of "hard" problems, you receive $6t$ points, where t is a number depending on the date when it is done. Passing all "hard" or all "ordinary" problems brings you $10t$ points. Solving of "**" (extra hard) problems is not obligatory, but each such problem gives you a credit for 2 "*" or "!" problems in the "hard" set.

The first 3 weeks after giving a handout, $t = 1.5$, between 21 and 35 days, $t = 1$, and afterwards, $t = 0.7$. The scores are not cumulative, only the best score for each handout counts.

10.1 L^2_p-metrics and differential operators

In this handout, the base manifold M is tacitly assumed compact.

Exercise 10.1. Let $|\cdot|^2_p$ be the usual L^2_p-metric on a torus T^n.

a. (*) Let $D := \sum_{P_\alpha} P_\alpha^2$ is the sum of all differential monomials of degree $\leqslant p$. Prove that the metric $|f|^2_p$ is equivalent to the metric $|f|^2_\bullet := \int_{T^n}(D(f), f)^2 \operatorname{Vol}$.

b. (!) Prove that this metric is equivalent to the metric

$$|f|^2_\circ := \int_{T^n} \left(1 + \sum_{i=1}^n \left|\frac{d^p f}{dt_i^p}\right|^2\right) \operatorname{Vol}$$

where $t_1, ..., t_n$ are coordinates on T^n.

c. (!) Prove that the map $f \longrightarrow \sum_i \frac{d^p f}{dt_i^p}$ from $L^2_p(T^n)$ to $L^2(T^n)$ is Fredholm, if p is even.

d. (!) Let $D_1: C^\infty T^n \longrightarrow C^\infty T^n$ be a differential operator which has the same symbol as $\sum_i \frac{d^p}{dt_i^p}$. Prove that D_1 defines a Fredholm map from $L^2_p(T^n)$ to $L^2(T^n)$, if p is even.

e. Let $\Delta: C^\infty T^n \longrightarrow C^\infty T^n$ be the Laplace operator, $\Delta = \sum_i \frac{d^2}{dt_i^2}$. Prove that Δ defines a Fredholm map from $L^2_2(T^n)$ to $L^2(T^n)$.

Exercise 10.2 (!). Let $D: B \longrightarrow B$ be a differential operator of order p such that the map $D: L^2_{p+i}(B) \longrightarrow L^2_i(B)$ is Fredholm, and D_1 an operator of order $< p$. Prove that $D + D_1$ is also Fredholm.

Hint. Use the Rellich lemma.

Exercise 10.3. Let M be a compact Riemannian manifold, equipped with a connection ∇, and B a vector bundle with metric and connection. Consider the iterated connection $B \stackrel{\nabla^p}{\longrightarrow} B \otimes \Lambda^1(M)^{\otimes p}$. Prove that $\oplus_{i=0}^p \nabla^i$ defines an isometric embedding of vector spaces $L^2_p(B) \longrightarrow \bigoplus_{i=0}^p L^2(B \otimes \Lambda^1(M)^{\otimes i})$.

Exercise 10.4. a. Consider the differential operator $b \longrightarrow (\nabla)^*\nabla b$. Prove that it has the same symbol as the Laplace operator Δ.

b. (!) In assumptions of the previous exercise, let $D: \ L^2_{2p}(B) \longrightarrow L^2(B)$ denote the differential operator $b \longrightarrow (\nabla^p)^*\nabla^p b$. Prove that the symbol of D is the same as the symbol of Δ^p.

Exercise 10.5. Let $D: \ B \longrightarrow B$ be a differential operator of order $2p$ such that its symbol is the same as of p-th power of the Laplace operator on a Riemannian manifold M. Prove that $D: \ L^2_{2p}(B) \longrightarrow L^2(B)$ is Fredholm

a. when (B, ∇) is trivial and M is a torus with flat metric

b. (!) when (B, ∇) is trivial and M is a torus with arbitrary metric

c. (!) on arbitrary Riemannian manifold, for any (B, ∇)

Hint. Use Exercise 10.3 and 10.4.

Exercise 10.6 (!). Let $D: \ B \longrightarrow B$ be a differential operator of order $2p$ with the same symbol as $(\nabla^p)^*\nabla^p$. Prove that $D: \ L^2_{2p+i}(B) \longrightarrow L^2_i(B)$ is Fredholm for all i.

Exercise 10.7 (*). Let $D: \ B \longrightarrow B$ be an elliptic operator of order p. Prove that $D: \ L^2_{p+i}(B) \longrightarrow L^2_i(B)$ is Fredholm for all i.

10.2 Green operator

Exercise 10.8. Let (B, ∇) be a bundle with connection over a Riemannian manifold M, and $\Delta: \ B \longrightarrow B$ an operator with the same symbol as $\nabla^*\nabla$, self-adjoint with respect to the L^2-metric.

a. Prove that $\ker \Delta$ is finite-dimensional, and $\operatorname{im} \Delta = \ker \Delta^\perp$ (orthogonal is taken with respect to the L^2-metric)

b. Prove that $\Delta\big|_{\operatorname{im}\Delta}$ is invertible in L^2_p-topology, for all p.

c. Prove thta $L^2(B) = \operatorname{im}\Delta \oplus \ker\Delta$.

d. (!) Prove that there exists **the Green operator**

$$G_\Delta : \ L^2_p(B) \longrightarrow L^2_p(B)$$

which is inverse to Δ on $\operatorname{im}\Delta$ and zero on $\ker\Delta$.

e. (!) Prove that $G_\Delta : \ L^2(B) \longrightarrow L^2(B)$ is compact and self-adjoint.

f. (!) Prove that $G_\Delta : \ L^2(B) \longrightarrow L^2(B)$ can be diagonalized in an orthonormal Hilbert basis on $L^2_p(B)$.

Exercise 10.9. Identify $L^2_p(B)$ with a subspace in $L^2_{p-i}(B)$, for any $i \geqslant 0$, using the continuous injective map $\mathsf{Id} : \ L^2_p(B) \longrightarrow L^2_{p-i}(B)$.

a. (!) Prove that $\bigcap_{p=0}^\infty L^2_p(B)$ is identified with the space $C^\infty B$ of smooth sections of B.

b. (*) Consider the topology on $\bigcap_p L^2_p(B)$ induced from all L^2_p (that is, a sequence $b_i \in \bigcap_p L^2_p(B)$ converges if it converges in L^2_p for all p). Denote this topology by L^2_∞. Prove that $b_i \in \bigcap_p L^2_p(B) = C^\infty B$ converges in L^2_∞ if and only if the sequence $\nabla^p b_i$ uniformly converges for any given p.

Exercise 10.10 (!). Let (B, ∇) be a bundle with connection over a Riemannian manifold M, and $\Delta : \ B \longrightarrow B$ an operator with the same symbol as $\nabla^*\nabla$, self-adjoint with respect to the L^2-metric. Prove that there exists an orthonormal basis in $L^2_0(B)$ diagonalizing B. Prove that all eigenvectors of Δ are smooth functions.

Hint. Use the previous exercise.

10.3 Harmonic forms

Exercise 10.11 (!). Let M be a Riemannian manifold, and $\Delta := dd^* + d^*d$ the Laplacian operators on the differential forms. Prove that Δ has the same symbol as $\nabla^*\nabla$, where $\nabla : \ \Lambda^*M \longrightarrow \Lambda^*M \otimes \Lambda^1 M$ is the connection on the bundle of all differential forms.

Definition 10.1. A differential form is called **harmonic** if it lies in $\ker\Delta$.

Exercise 10.12. Prove that $\ker\Delta = \ker d\cap \ker d^*$.

Hint.

$$(\Delta\eta,\eta) = (dd^*\eta,\eta) + (d^*d\eta,\eta) = (d\eta,d\eta) + (d^*\eta,d^*\eta)$$

Exercise 10.13. Prove that $\operatorname{im}\Delta = \operatorname{im} d + \operatorname{im} d^*$.

Exercise 10.14 (!). Prove that the image of d and d^* is closed in L^2-topology on $\Lambda^*(M)$. Prove that $\ker d^* = (\operatorname{im} d)^\perp$ and $\ker d = (\operatorname{im} d^*)^\perp$

Exercise 10.15. Prove that $\Lambda^*(M) = \operatorname{im} d \oplus \operatorname{im} d^* \oplus \ker\Delta$.

Exercise 10.16 (!). Prove that $\ker d = \operatorname{im} d \oplus \ker\Delta$.

Remark 10.1. From the previous exercise it follows that de Rham cohomology of a (compact) manifold are identified with the space of hermonic forms.

Exercise 10.17 (*). Let M be a Riemannian manifold (not necessarily compact), and $\Lambda^i_0(M)$ be the sheaf of harmonic i-form. Let $\mathbb{R}_M$ denote the constant sheaf. Prove that the complex of sheaves

$$0 \longrightarrow \mathbb{R}_M \hookrightarrow \Lambda^0_0(M) \xrightarrow{d} \Lambda^1_0(M) \xrightarrow{d} \Lambda^2_0(M) \xrightarrow{d}$$

is exact.

Exercise 10.18. Let M be a compact Riemannian manifold with boundary.

a. (*) Prove that $\Delta: C^\infty M \longrightarrow C^\infty M$ is surjective.

b. (**) Prove that $\Delta: \Lambda^i M \longrightarrow \Lambda^i M$ is surjective for all i, or find a counterexample.