Hodge theory 11: exponential map and Frobenius theorem

Rules: You may choose to solve only "hard" exercises (marked with !, * and **) or "ordinary" ones (marked with ! or unmarked), or both, if you want to have extra stuff to work. To have a perfect score, a student must obtain (in average) a score of 10 points per week.

If you have got credit for 2/3 of ordinary problems or 2/3 of "hard" problems, you receive $6t$ points, where t is a number depending on the date when it is done. Passing all "hard" or all "ordinary" problems brings you $10t$ points. Solving of "**" (extra hard) problems is not obligatory, but each such problem gives you a credit for 2 "*" or "!" problems in the "hard" set.

The first 3 weeks after giving a handout, $t = 1.5$, between 21 and 35 days, $t = 1$, and afterwards, $t = 0.7$. The scores are not cumulative, only the best score for each handout counts.

11.1 Exponential map

Definition 11.1. Let $\nabla: TM \longrightarrow TM \otimes \Lambda^1 M$ be a connection. **Geodesic** with respect to ∇ is a map $\gamma: [a,b] \longrightarrow M$ which satisfies $\nabla_{\dot\gamma}(\dot\gamma) = 0$, where $\dot\gamma(t) = \frac{\gamma(u)}{du}\Big|_{u=t} \in TM\Big|_{\operatorname{im}\gamma}$ is the derivative of γ, and the equation $\nabla_{\dot\gamma}(\dot\gamma)$ is understood as parallelism of $\dot\gamma$, considered as a section of the vector bundle $TM\Big|_{\operatorname{im}\gamma}$ with connection ∇.

Exercise 11.1. Let $v \in T_xM$.

a. (!) Prove that for ε sufficiently small, there exists a geodesic $\gamma: [0,\varepsilon] \longrightarrow M$ such that $\gamma(0) = x$ and $\dot\gamma(0) = v$.

b. (!) Prove that geodesic is unique.

Definition 11.2. Let $U \subset T_xM$ be the set of tangent vectors such that for any $v \in U$ there exists a geodesic $\gamma_v: [0,1] \longrightarrow M$, such that $\gamma_v(0) = x$ and $\dot\gamma_v(0) = v$. The map $v \longrightarrow \gamma(1)$ is called **exponential map**.

Exercise 11.2 (!). Prove that exponential map defines a diffeomorphism from a neighbourhood of 0 in U to M.

Definition 11.3. The corresponding coordinates are called **geodesic coordinates**, or **normal coordinates**.

Exercise 11.3. Suppose that M is a Riemannian manifold, and ∇ an orthogonal connection. Prove that the exponential map exp is defined on all T_xM, that is, for any vector $v \in T_xM$ there exists a geodesic map $\gamma: [0,\infty[$ tangent to v

a. (!) when M is compact

b. (*) when it is complete as a metric space.

c. (**) Suppose that exp is defined on all T_xM. Prove that M is complete as a metric space.

Exercise 11.4 (*). Find a compact manifold and a connection such that the exponential map is not defined on the whole T_xM (that is, for some vector $v \in T_xM$ there is no geodesic map $\gamma: [0,1]$ tangent to v).

11.2 Jacobi fields

Definition 11.4. Let M be a manifold with connection, and $\exp: T_xM \longrightarrow M$ the exponential map (which can be defined on the whole T_xM or locally in a neighbourhood U of 0). Let $A \in \operatorname{End}(T_xM)$ be a matrix, and $a \in T(T_xM)$ a linear vector field $x \longrightarrow A(x)$ associated with A. **Jacobi field** is a vector field on $\exp(U)$ obtained as an image of a under the exponential map.

Exercise 11.5 (!). Let γ_0 be a geodesic passing through $x \in M$, and $J \in TM\Big|_{\gamma_0}$ a Jacobi field. Prove that there exists a smooth family γ_t of geodesics passing through x such that $J\Big|_{\gamma_0(r)} = \frac{\gamma_t(r)}{dt}$.

Hint. A flow of diffeomorphisms associated with a Jacobi field maps a geodesic passing through x to another geodesic passing through x.

Remark 11.1. Usually, one defines Jacobi field as a vector field $J \in TM\Big|_{\gamma_0}$ on a geodesic such that $J\Big|_{\gamma_0(r)} = \frac{\gamma_t(r)}{dt}$ for some smooth family γ_t of geodesics. As shown above, this definition is equivalent to ours.

Definition 11.5. Radial vector field on a vector space $V = \mathbb{R}^n$ is the linear vector field associated with the map $x \longrightarrow x$. In standard coordinates it can be written as $\sum x_i \frac{d}{dx_i}$.

Exercise 11.6. Let X be a vector field defined in a star-shaped neighbourhood of 0 in $\mathbb{R}^n$. Prove that X is a linear vector field, if

a. X commutes with the radial vector field R

b. $[[X, R], R] = 0$.

Hint. Use the Taylor series for X and the following observation: for any homogeneous vector field R_d of degree d, one has $[R, X] = dX$.

Exercise 11.7. Let $\exp: T_xM \longrightarrow M$ be the exponential map, defining a diffeomorphism in a star-shaped neighbourhood $U \subset T_xM$, and $\theta \in TM$ the image of the radial vector field under exp. Prove that J is a Jacobi vector field if and only if

a. (!) $[J, \theta] = 0$

b. (!) $[[J, \theta], \theta] = 0$.

Remark 11.2. Abusing the language, we call the vector field $\theta \in T\exp(U)$ defined above "the radial vector field" as well.

Definition 11.6. Let $\nabla : TM \longrightarrow TM \otimes \Lambda^1 M$ be a connection. **Torsion** of ∇ is a map $T_\nabla : \Lambda^2 TM \longrightarrow TM$ mapping vector fields $X, Y \in TM$ to $\nabla_X Y - \nabla_Y X - [X, Y]$. A connection is called **torsion-free** if its torsion vanishes. **Curvature** of ∇ is a map $\Theta_\nabla : \Lambda^2 TM \longrightarrow \operatorname{End}(TM)$ mapping vector fields $X, Y \in TM$ to $[\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ considered as a map from TM to TM.

Exercise 11.8. Prove that torsion and curvature are $C^\infty(M)$-linear.

Exercise 11.9. Let U be an image of a star-shaped domain $W \subset T_x M$ under the exponential map $\exp : T_x M \longrightarrow M$ which is a diffeomorphism on W. Denote by $\theta \in TU$ the radial vector field, and let $J \in TU$ be another vector field. Assume that the connection ∇ is torsion-free.

a. Prove that J is Jacobi if and only if $\nabla_J \theta = \nabla_\theta J$. Prove that J is Jacobi if and only if

$$\nabla_{[\theta, J]} \theta = \nabla_\theta (\nabla_\theta J - \nabla_J \theta). \tag{11.1}$$

b. Prove that θ is a radial vector field if and only if $\nabla_\theta \theta = \theta$.

c. Prove that

$$-\nabla_{[\theta, J]} = \Theta_\nabla(\theta, J) - \nabla_\theta \nabla_J + \nabla_J \nabla_\theta$$

and

$$-\nabla_{[\theta, J]} \theta = \Theta_\nabla(\theta, J)(\theta) - \nabla_\theta \nabla_J \theta + \nabla_J \theta.$$

Use this to rewrite (11.1) as

$$\Theta_\nabla(\theta, J)(\theta) = \nabla_\theta \nabla_\theta(J) - \nabla_J \theta. \tag{11.2}$$

Exercise 11.10 (!). Let $J \in TM\Big|_\gamma$ be a vector field tangent to the geodesic. Denote by $\ddot{J}$ the second derivative of J along γ, with $\ddot{J} = \nabla_{\dot\gamma} \nabla_{\dot\gamma}(J)$. Prove that J is a Jacobi field if and only if $\ddot{J} = \Theta_\nabla(\dot\gamma, J, \dot\gamma)$

Hint. Let t be a parametrization of a geodesic $\gamma(t)$. Prove that θ is tangent to γ and $t\dot\gamma = \theta$ on γ, and deduce that

$$\nabla_\theta \nabla_\theta = t^2 \nabla_{\dot\gamma} \nabla_{\dot\gamma} + t \nabla_{\dot\gamma}.$$

using (11.2).

Remark 11.3. $\ddot{J} = \Theta_\nabla(\dot\gamma, J, \dot\gamma)$ is a second order ODE.

11.3 Torsion and Frobenius theorem

Definition 11.7. Let $B \subset TM$ be a sub-bundle of tangent bundle, and $\Phi : \Lambda^2 B \longrightarrow TM/B$ maps vector fields $b_1, b_2 \in B \subset TM$ to their commutator $[b_1, b_2]$ mod B. Then Φ is called **Frobenius form**.

Exercise 11.11. Prove that the Frobenius form is $C^\infty M$-linear.

Exercise 11.12. Let $B \subset TM$ be a sub-bundle. Prove that there exists a connection $\nabla : TM \longrightarrow TM \otimes \Lambda^1 M$ such that $\nabla(B) \subset B \otimes \Lambda^1 M$.

Remark 11.4. In this case we say that **connection ∇ preserves** B.

Exercise 11.13. Suppose that a connection $\nabla : TM \longrightarrow TM \otimes \Lambda^1 M$ preserves a sub-bundle $B \subset TM$.

a. Prove that T_∇ maps $\Lambda^2 B$ to B if and only if its Frobenius form vanishes.

b. (!) Suppose that the Frobenius form of B vanishes. Prove that there exists a torsion-free connection preserving $B \subset TM$

Hint. Use the same argument as used in the proof of existence of Levi-Civita connection.

Exercise 11.14. Let $B \subset TM$ be a sub-bundle, preserved by a connection ∇, and Θ_∇ its curvature. Prove that $\Theta_\nabla(X, Y, Z) \in B$ whenever $Z \in B$.

Exercise 11.15. Let $B \subset TM$ be a sub-bundle, preserved by a torsion-free connection ∇.

a. Denote by θ the radial vector field, defined in a neighbourhood of $x \in M$. Prove that for any Jacobi field J such that $J\Big|_x \in B$ and $\nabla_\theta J\Big|_x \in B$, one has $J \in B$.

b. (!) Use his to show that the exponential map maps the subspace $B\Big|_x \subset T_x M$ to a submanifold of M tangent to B.

Exercise 11.16 (!). Prove Frobenius theorem: given $B \subset TM$ such that $[B, B] \subset B$, for any $x \in M$ there exists a neighbourhood $U \subset M$ and a family of submanifolds $V_t \subset U$ parametrized by $t \in Z$ such that U is obtained as a disjoint union of V_t, and $TV_t = B$ at each point of V_t for each $t \in Z$.

Remark 11.5. In these assumptions, submanifolds V_t are called **leaves of the foliation** defined by B, and B is called **an involutive** or **holonomic** sub-bundle of TM.