Hodge theory 12: Complex and Kähler structures on symmetric spaces

Rules: You may choose to solve only "hard" exercises (marked with !, * and **) or "ordinary" ones (marked with ! or unmarked), or both, if you want to have extra stuff to work. To have a perfect score, a student must obtain (in average) a score of 10 points per week.

If you have got credit for 2/3 of ordinary problems or 2/3 of "hard" problems, you receive $6t$ points, where t is a number depending on the date when it is done. Passing all "hard" or all "ordinary" problems brings you $10t$ points. Solving of "**" (extra hard) problems is not obligatory, but each such problem gives you a credit for 2 "*" or "!" problems in the "hard" set.

The first 3 weeks after giving a handout, $t = 1.5$, between 21 and 35 days, $t = 1$, and afterwards, $t = 0.7$. The scores are not cumulative, only the best score for each handout counts.

12.1 Almost complex, Hermitian and Kähler structures

Definition 12.1. Let M be a manifold. An endomorphism $I \in \operatorname{End}(TM)$, $I^2 = -\operatorname{Id}_{TM}$ is called **an almost complex structure**, and its $\sqrt{-1}$-eigenbundle is denoted as $T^{1,0}M \subset TM \otimes \mathbb{C}$. An almost complex structure I is called **integrable** if $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$. In this case (M, I) is called **a complex manifold**. A Riemannian metric on an almost complex manifold is called **Hermitian** if it is I-invariant.

Exercise 12.1. Let $U = V \oplus W$ be vector spaces. Prove that their Grassmann algebras are decomposed as follows: $\Lambda^n(U) = \bigoplus_{p+q=n} \Lambda^p V \otimes \Lambda^q W$.[1]

Definition 12.2. Consider the eigenvalue decomposition $\Lambda^1(M,\mathbb{C}) = \Lambda^{1,0}(M) \oplus \Lambda^{0,1}(M)$ associated with the action of I, with $I\Big|_{\Lambda^{1,0}(M)} = \sqrt{-1}$ and $I\Big|_{\Lambda^{0,1}(M)} = -\sqrt{-1}$. It induces the decomposition on the de Rham algebra

$$\Lambda^k(M,\mathbb{C}) = \bigoplus_{p+q=k} \Lambda^p(\Lambda^{1,0}(M)) \otimes \Lambda^q(\Lambda^{0,1}(M))$$

as shown above. The bundles $\Lambda^p(\Lambda^{1,0}(M))$ and $\Lambda^q(\Lambda^{0,1}(M))$ are denoted $\Lambda^{p,0}(M)$ and $\Lambda^{0,q}(M)$, and the component $\Lambda^p(\Lambda^{1,0}(M)) \otimes \Lambda^q(\Lambda^{0,1}(M))$ is denoted $\Lambda^{p,q}(M)$. The decomposition $\Lambda^k(M,\mathbb{C}) = \bigoplus_{p+q=k} \Lambda^{p,q}(M)$ is called **the Hodge decomposition**, the sections of $\Lambda^{p,q}(M)$ are called (p,q)-forms.

Exercise 12.2. Let (M, I) be an almost complex manifold, and h an I-invariant Riemannian form.

a. Prove that $\omega(x,y) = h(Ix,y)$ is a (1,1)-form.

b. (!) Prove that any Hermitian form h is obtained from a (1,1)-form ω such that $\omega(x, Ix) > 0$ for all non-zero tangent vectors $x \in T_mM$.

Exercise 12.3. Prove that any almost complex manifold admits a Hermitian metric.

[1] This decomposition is not multiplicative.

Exercise 12.4 (*). Let M be a manifold admitting a non-degenerate 2-form. Prone that M admits an almost complex structure.

Definition 12.3. Let (M, I, h) be an almost complex Hermitian manifold The form $\omega(x, y) = h(Ix, y)$ is called **the fundamental form** of M. The triple (M, I, ω) is called **a Kähler triple** if I is integrable and ω is closed. In this case M is called **the Kähler manifold**, h **the Kähler metric** and ω **the Kähler form.**

Remark 12.1. Recall that **symplectic manifold** is a manifold equipped with a non-degenerate, closed 2-form. Clearly, the Kähler form is closed and non-degenerate.

Exercise 12.5. Find a complex, compact manifold not admitting a Kähler metric.

Exercise 12.6 (**). Find a complex, compact manifold not admitting a Kähler metric, but admitting a symplectic structure.

12.2 Symmetric spaces

Definition 12.4. Homogeneous space is a manifold with transitive action of a Lie group (often assumed connected).

Exercise 12.7. Let M be a connected manifold with transitive action of a Lie group G, and H be a stabilizer of a point $x \in M$ (in this case, H is called **the isotropy group** of x).

a. Prove that M is identified with the space of orbits G/H.

b. (!) Let $x, y \in M$, and H_x, H_y be the corresponding isotropy groups. Prove that H_x and H_y are conjugate by some element of G.

Exercise 12.8. Let $M = G/H$ be a homogeneous space with compact H. Assume that M is connected and all non-unit $g \in G$ act non-trivially.

a. (!) Prove that M admits a G-invariant Riemannian structure.

b. (*) Prove that the natural map from the isotropy group of x to $GL(T_xM)$ is injective.

c. (**) Is it always injective if H is not necessarily compact?

Definition 12.5. A tensor on a manifold M is a section of the **tensor bundle** $TM^{\otimes p} \otimes T^*M^{\otimes q}$. Whenever G acts on M by diffeomorphisms, it acts on the space of tensors, because tensors are functorial.

Exercise 12.9 (!). Let $M = G/H$ be a homogeneous space, and H_x the isotropy group of $x \in M$. Construct a bijective correspondence between G-invariant tensors on M and H_x-invariant vectors in $T_xM^{\otimes p} \otimes T_x^*M^{\otimes q}$.

Definition 12.6. A homogeneous space $M = G/H$ is called **symmetric space** if M admits a G-invariant Riemannian metric and H_x contains an involution ι which acts as $-\operatorname{Id}$ on T_xM.

Definition 12.7. $SO(n)$ denotes the **special orthogonal** group (the group of all orthogonal matrices preserving the orientation). $U(n)$ is **unitary group** (the group of all complex-linear matrices preserving a Hermitian form). $SU(n)$ is intersection of $U(n)$ and $SL(n,\mathbb{C})$.

Exercise 12.10. Consider the spaces S^{2n}, $\mathbb{C}P^n$, $\mathbb{H}P^n$ equipped with the natural action of $SO(2n+1)$, $U(n+1)$ and $Sp(n+1) := GL(n+1,\mathbb{H}) \cap SO(4n+4)$. Prove that they are symmetric spaces.

Exercise 12.11. Consider the Grassmannian $\operatorname{Gr}_{\mathbb{R}}(p,q) := \frac{SO(p+q)}{SO(p)\times SO(q)}$,

a. (!) Prove that it is a symmetric space when p or q is even.

b. (*) Prove it for all p, q.

Definition 12.8. An **odd tensor** on a symmetric space is a tensor $\Psi \in TM^{\otimes p} \otimes T^*M^{\otimes q}$ for $p + q$ odd.

Exercise 12.12. Let $M = G/H$ be a symmetric space, and Ψ a G-invariant odd tensor. Prove that $\Psi = 0$.

Exercise 12.13. Let $M = G/H$ be a symmetric space, and ω a G-invariant differential form.

a. Prove that $d\omega = 0$ and ω is even.

b. (!) Suppose that M is equipped with a G-invariant Riemannian form. Prove that ω is harmonic.

c. (!) Assume in addition that M is compact and G is connected. Prove that any harmonic form is G-invariant.

12.3 Kähler structures on symmetric spaces

Exercise 12.14 (!). Let $M = G/H$ be a symmetric space, and I a G-invariant almost complex structure. Prove that I is integrable.

Exercise 12.15 (!). $M = G/H$ be a symmetric space, I a G-invariant almost complex structure, and h a G-invariant Hermitian form. Prove that (M, I, h) is Kähler.

Exercise 12.16. Construct a structure of symmetric space and a G-invariant complex structure on the following spaces.

a. $\mathbb{C}P^n$ (also prove that it is Kähler).

b. (!) $\operatorname{Gr}_{\mathbb{R}}(2,n) := \frac{SO(n+2)}{SO(n)\times SO(2)}$

Exercise 12.17 (!). Let $M = G/H$ be a homogeneous space such that the isotropy group H_x acts on the (real) projectivization $\mathbb{P}T_xM$ transitively. Prove that the G-invariant Riemannian metric on M is unique up to a constant multiplier.

Exercise 12.18 (*). Consider the Grassmannian space $\operatorname{Gr}_{\mathbb{R}}(p,q) := \frac{SO(p+q)}{SO(p)\times SO(q)}$. Prove that $\operatorname{Gr}_{\mathbb{R}}(p,q)$ admits a $SO(p+q)$-invariant metric. Prove that this metric is unique up to a constant multiplier, when $p > 2$ or $q > 2$.

Exercise 12.19. Construct a $U(n+1)$-invariant Hermitian metric on $\mathbb{C}P^n$ (it is called **Fubini-Study metric**).

a. Prove that this metric is unique up to a constant.

b. Prove that it is Kähler.

Definition 12.9. $U(p,q)$ is the group of all complex-linear matrices preserving a pseudo-Hermitian metric h of signature (p,q), with $h(x_1,...,x_{p+q}) = \sum_{i=1}^p |x_i|^2 - \sum_{i=q+1}^{p+q} |x_j|^2$.

Exercise 12.20. a. (!) Construct a $U(1,n)$-invariant metric and complex structure on $M := \frac{U(1,n)}{U(1)\times U(n)}$.

b. (!) Prove that it is Kähler.

c. (*) Prove that M is biholomorphic to an open ball in $\mathbb{C}^n$.

d. (**) Prove that all complex automorphisms of an open ball are isometries with respect to this metric.

Remark 12.2. This metric on an open ball is called **Bergman metric**, or **complex hyperbolic metric**.

Exercise 12.21 (*). Construct an $SO(n+2)$-invariant Kähler structure on the Grassmannian $\operatorname{Gr}_{\mathbb{R}}(2,n) := \frac{SO(n+2)}{SO(n)\times SO(2)}$.