Hodge theory 13: Foliations, fiber bundles, and dd^c

Rules: You may choose to solve only "hard" exercises (marked with !, * and **) or "ordinary" ones (marked with ! or unmarked), or both, if you want to have extra stuff to work. To have a perfect score, a student must obtain (in average) a score of 10 points per week.

If you have got credit for 2/3 of ordinary problems or 2/3 of "hard" problems, you receive $6t$ points, where t is a number depending on the date when it is done. Passing all "hard" or all "ordinary" problems brings you $10t$ points. Solving of "**" (extra hard) problems is not obligatory, but each such problem gives you a credit for 2 "*" or "!" problems in the "hard" set.

The first 3 weeks after giving a handout, $t = 1.5$, between 21 and 35 days, $t = 1$, and afterwards, $t = 0.7$. The scores are not cumulative, only the best score for each handout counts.

13.1 Foliations

Definition 13.1. Sheaf of submanifolds on M is a sheaf $\mathcal{F}$ of sets mapping each U to a collection of its closed submanifolds, with restriction maps $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ mapping each submanifold $Z \in \mathcal{F}(U)$ to $Z \cap V$. **A foliation** is a sheaf of submanifolds $\mathcal{F}$ on M such that each $x \in M$ has a neighbourhood U which is decomposed onto a product $U = A \times B$, with $\mathcal{F}(U)$ being all fibers of the projection $U \longrightarrow B$. **A leaf** of the foliation $\mathcal{F}$ is a connected smooth manifold Z immersed to M in such a way that any closed connected component of $Z \cap U$ is an element of $\mathcal{F}(U)$. **Closed leaf** is a leaf with closed image.

Exercise 13.1. Let $\mathcal{F}$ be a foliation on M. Prove that there exists a continuous map $\pi : M \longrightarrow Z$ with all leaves of $\mathcal{F}$ obtained as $\pi^{-1}(z)$ for some $z \in Z$ and $U \subset Z$ open if and only if $\pi^{-1}(U)$ is open.

Definition 13.2. In this case Z is called **the leaf space** of $\mathcal{F}$.

Exercise 13.2.

a. (!) Let $\mathcal{F}$ be a foliation on M with all leaves compact. Prove that in this case the leaf space of $\mathcal{F}$ is Hausdorff.

b. (**) Is this true for all foliations with closed leaves?

Exercise 13.3. Find a foliation with all leaves dense.

Exercise 13.4 (!). Find a foliation with all leaves closed, but not all of them diffeomorphic.

Exercise 13.5 (!). Let $\mathcal{F}$ be a foliation with compact leaves on a compact manifold M. Prove that its leaf space is smooth, or find a counterexample.

Definition 13.3. A foliation on M is called **fiber bundle** if all its leaves are closed and the projection $M \longrightarrow Z$ to its leaf space is locally trivial.

Exercise 13.6 (!). Let (M, ω) be a compact symplectic manifold and $\pi : M \longrightarrow Z$ a fiber bundle. Assume that ω restricted to fibers of π vanishes (in this case the fibers are called **Lagrangian submanifolds**, and π **a Lagrangian fibration**). Prove that all fibers of π have trivial tangent bundle.

Exercise 13.7. Let $\mathcal{F}$ be a foliation on M, and $T\mathcal{F}$ the sheaf of all vector fields tangent to leaves of $\mathcal{F}$.

a. Prove that $T\mathcal{F} \subset TM$ is a sub-bundle of TM.

b. (!) Prove that the sub-bundle $T\mathcal{F} \subset TM$ uniquely determines the foliation $\mathcal{F}$.

c. (!) Prove that $[T\mathcal{F}, T\mathcal{F}] \subset T\mathcal{F}$.

d. (!) (Frobenius theorem) Prove that any sub-bundle $B \subset TM$ such that $[B, B] \subset B$ is tangent to a certain foliation determined by B.

Hint. To prove the Frobenius theorem, use the exercises from Handout 11.

13.2 Basic forms

Definition 13.4. Let $\mathcal{F}$ be a foliation and $B = T\mathcal{F}$ its tangent bundle. A differential form $\eta \in \Lambda^* M$ is called **basic** with respect to B if for all vector fields $X \in B$, one has $\operatorname{Lie}_X \eta = 0$ and $\eta \lrcorner X = 0$.

Exercise 13.8. Prove that a closed form is basic if $\eta \lrcorner X = 0$ for all $X \in B$.

Exercise 13.9. Let $X_1, ... X_n \subset B$ be vector fields generating B over $C^\infty M$, and η a differential form such that $\operatorname{Lie}_{X_i} \eta = 0$ and $\eta \lrcorner X_i = 0$ for all i. Prove that η is basic.

Remark 13.1. This exercise is non-trivial, because the Lie derivative $\operatorname{Lie}_X \eta$ is not C^∞-linear in X.

Exercise 13.10. Let $\pi : M \longrightarrow Z$ be a differentiable map of smooth manifolds with differential surjective everywhere (further on, such maps will be called **smooth maps**). Prove that π is **open**, that is, the image $\pi(U)$ of an open set is always open.

Definition 13.5. Let $\pi : M \longrightarrow Z$ be a smooth map and E a vector bundle on Z, considered as a locally free sheaf of $C^\infty Z$-modules. Consider the sheaf-theoretic pullback $\pi^\bullet E$, with sections of $\pi^\bullet E$ over an open subset $U \subset Z$ given by $\pi^\bullet E(U) = E(\pi(U))$. **The pullback** of the vector bundle E is $\pi^\bullet E \otimes_{\pi^\bullet C^\infty Z} C^\infty M$. It is not hard to see that this is also a vector bundle, of the same rank as E.

Exercise 13.11. Let Z be the leaf space of the foliation $\mathcal{F}$; we assume that the projection $\pi : M \longrightarrow Z$ is smooth.

a. Let $\Lambda^*_\pi(M)$ be the bundle of all forms $\eta \in \Lambda^* M$ such that $\eta \lrcorner X = 0$. Prove that $\Lambda^*_\pi(M) = \pi^* \Lambda^*(M)$.

b. For any bundle E on Z, represent the sections of $\pi^* E$ by linear combinations of $f \otimes \pi^\bullet e$, where $f \in C^\infty M$, e is a section of E, and $\pi^\bullet e$ the corresponding section of $\pi^\bullet E$. For any $X \in B$, define $\operatorname{Lie}_X(f \otimes \pi^\bullet e) := \operatorname{Lie}_X(f) \otimes \pi^\bullet e$. Prove that this map satisfies the Leibitz rule. Prove that for all sections $e \in E$, one has
$$e \in \pi^\bullet E \Leftrightarrow \operatorname{Lie}_X e = 0 \quad \forall X \in B.$$

c. (!) Prove that a form η is basic if and only if η lies in $\pi^\bullet \Lambda^*(Z) \subset \Lambda^*_\pi(M) \subset \Lambda^*(M)$.

d. (!) Prove that the space of basic forms on M is naturally isomorphic to $\Lambda^* Z$.

Exercise 13.12 (**). For any given $0 < i < n$ find a foliation $\mathcal{F}$ of codimension n on a compact manifold M such that all basic i-forms vanish.

Exercise 13.13 (*). Denote basic forms by $\Lambda^*_B(M)$. Prove that the de Rham differential of a basic form is again basic. **Basic cohomology** is the quotient $\frac{\ker d\big|_{\Lambda^*_B(M)}}{d(\Lambda^*_B(M))}$. Prove that the space of basic cohomology of a compact manifold is finite-dimensional, for any foliation.

13.3 The twisted differential d^c

Definition 13.6. Let (M, I) be an almost complex manifold and $T^{1,0}(M) \oplus T^{0,1}(M)$ the Hodge decomposition. Consider the corresponding Frobenius form $N \in \operatorname{Hom}(\Lambda^2(T^{1,0}(M)), T^{0,1}(M))$. This map is called **the Nijenhuis tensor**. An almost complex structure is called **(formally) integrable** if its Nijenhuis tensor vanishes.

Remark 13.2. Please solve the exercises in this subsection without the use of Newlander-Nirenberg theorem.

Exercise 13.14. Let M be an almost complex manifold, and
$$d^{i,j} : \Lambda^{p,q}(M) \longrightarrow \Lambda^{p+i,q+j}(M), \quad i + j = 1$$
be the Hodge component of de Rham differential.

a. (!) Prove that $d^{i,j} = 0$ unless $(i, j) = (0, 1), (1, 0), (2, -1)$ or $(-1, 2)$.

b. (!) Prove that the operators $d^{2,-1}$ and $d^{-1,2}$ are $C^\infty(M)$-linear.

c. (!) Prove that $d^{2,-1} = d^{-1,2} = 0$ when the almost complex structure on M is integrable.

d. (*) Prove that the map $d^{2,-1} : \Lambda^{0,1}(M) \longrightarrow \Lambda^{2,0}(M)$ is dual to the Nijenhuis tensor.

Definition 13.7. We extend I to differential forms multiplicatively, $I\Big|_{\Lambda^{p,q}(M)} = (\sqrt{-1})^{p-q}$. Let $d^c := IdI^{-1}$. This operator is called **the twisted differential**.

Exercise 13.15 (!). Prove that I is integrable if and only if $dd^c = -d^c d$.

Exercise 13.16 (!). Prove that I is integrable if and only if $d^{1,0} = \frac{d-\sqrt{-1}d^c}{2}$. Prove that in this case $d^{0,1} = \frac{d+\sqrt{-1}d^c}{2}$.

Definition 13.8. The operators $d^{1,0}$, $d^{0,1}$ on a complex manifold are denoted $\partial : \Lambda^{p,q}(M) \longrightarrow \Lambda^{p+1,q}(M)$ and $\bar\partial : \Lambda^{p,q}(M) \longrightarrow \Lambda^{p,q+1}(M)$.

Exercise 13.17. Prove that $-2\sqrt{-1}\partial\bar\partial = dd^c$.

Definition 13.9. The operator dd^c is called **the pluri-Laplacian**, and function f with $dd^c(f) = 0$ is called **pluri-harmonic**.

Exercise 13.18. Let B be an open subset in $\mathbb{C}$ and $\omega = dx \wedge dy$ the standard volume form. Prove that $dd^c(f) = \Delta(f)\omega$, where Δ is the Laplacian.

Definition 13.10. Recall that a $\mathbb{C}$-valued function f on a complex manifold is called **anti-holomorphic** if $\bar f$ is holomorpic.

Exercise 13.19. Let f be a sum of a holomorphic and an anti-holomorphic function is pluri-harmonic.

Exercise 13.20 (!). Let f be a smooth real function on $\mathbb{C}^n$. Suppose that f is pluri-harmonic. Prove that restriction of f to any complex curve $Z \subset \mathbb{C}^n$ is a real part of a holomorphic function.

Exercise 13.21. Let f be a real pluri-harmonic function on a poly-disc U in $\mathbb{C}^n$.

a. (*) Using the Poincaré-Dolbeault-Grothendieck lemma, prove that f is a real part of a holomorphic function.

b. (*) Deduce from the Poincaré-Dolbeault-Grothendieck lemma that a function is pluri-harmonic if and only if it is represented locally as a sum of a holomorphic and antiholomorphic function.

Exercise 13.22 (!). Find a complex manifold M and a real pluriharmonic function f which is not a real part of a holomorphic function on M.