Hodge theory 14: Fubini-Study form

Rules: You may choose to solve only "hard" exercises (marked with !, * and **) or "ordinary" ones (marked with ! or unmarked), or both, if you want to have extra stuff to work. To have a perfect score, a student must obtain (in average) a score of 10 points per week.

If you have got credit for 2/3 of ordinary problems or 2/3 of "hard" problems, you receive $6t$ points, where t is a number depending on the date when it is done. Passing all "hard" or all "ordinary" problems brings you $10t$ points. Solving of "**" (extra hard) problems is not obligatory, but each such problem gives you a credit for 2 "*" or "!" problems in the "hard" set.

The first 3 weeks after giving a handout, $t = 1.5$, between 21 and 35 days, $t = 1$, and afterwards, $t = 0.7$. The scores are not cumulative, only the best score for each handout counts.

14.1 Plurisubharmonic functions

Definition 14.1. A real-valued function f on a compact manifold is called **strictly plurisubharmonic** if $dd^c f$ is a Kähler form.

Exercise 14.1. Consider a function l on $\mathbb{C}^n$, $l(z_1, ..., z_n) = \sum |z_i|^2$. Prove that $dd^c l$ is the standard Kähler form on $\mathbb{C}^n$.

Exercise 14.2. Let ω be a Kähler form on a polydisk. Using the Poincaré-Dolbeault-Grothendieck lemma, prove that $\omega = dd^c(f)$ for some strictly plurisubharmonic function f.

Exercise 14.3 (**). Let f a plurisubharmonic funcction on $\mathbb{C}$. Prove that f cannot be bounded.

Exercise 14.4 (*). Let f be a plurisubharmonic function on M. Prove that f has does not have a maximum.

Exercise 14.5. Prove that $dd^c f(\phi) = f' dd^c\phi + f'' d\phi \wedge d^c\phi$. for any real-valued function ϕ on a complex manifold and any smooth $f: \mathbb{R} \longrightarrow \mathbb{R}$.

Exercise 14.6 (!). Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be a convex smooth function with $f' > 0$ everywhere. Prove that $f(\phi)$ is strictly plurisubharmonic whenever ϕ is strictly plurisubharmonic.

Exercise 14.7. Let $f: \mathbb{R}^2 \longrightarrow R$ be a convex function with $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ positive everywhere, and ψ, ψ strictly plurisubharmonic functions.

a. (!) Prove that the function $f(\phi, \psi)$ is also strictly plurisubharmonic.

b. (*) Prove that for any $\varepsilon > 0$ there exists a strictly plurisubharmonic function ξ such that $\xi = \max(\phi, \psi)$ whenever $|\phi - \psi| > \varepsilon$

Exercise 14.8. Prove that $dd^c \log \phi = \frac{dd^c\phi}{\phi} - \frac{d\phi \wedge d^c\phi}{\phi^2}$. for any real-valued function ϕ on a complex manifold.

Exercise 14.9. For any real function f, the form $dd^c f = -\sqrt{-1}2\partial\bar\partial f$ is of type (1,1), hence the form $h_f := dd^c f(I(\cdot), \cdot)$ is symmetric and pseudo-Hermitian. For any Hermitian form s on M, prove that in a neighbourhood of each point there exists an orthonormal basis such that h_f is diagonal. Let $\alpha_1, ..., \alpha_n$ be the eigenvalues of $h_f s^{-1}$.

a. (*) Prove that the map $M \longrightarrow \mathbb{R}^n/\Sigma_n$ mapping $x \in M$ to the unordered collection of all eigenvalues of h_f is continuous.

b. (!) Prove that the sign of these eigenvalues at $x \in M$ is independent from the choice of s.

Definition 14.2. We say that $dd^c f$ **has positive/negative/zero eigenvalues** when $h_f s^{-1}$ has positive (negative, zero) eigenvalues for some (hence, any) Hermitian forms s on M.

14.2 Fubini-Study form

Exercise 14.10. Let $l(z_1, ..., z_n) = \sum |z_i|^2$ be the function on $\mathbb{C}^n$ defined above. Prove that $dd^c \log l = \frac{dd^c l}{l} - \frac{dl \wedge d^c l}{l^2}$. Prove that for $n > 1$, the form $dd^c \log l$ has at least one positive eigenvalue.

Exercise 14.11. a. Consider the function $|z|^2 = z\bar z$ on $\mathbb{C}^*$, and let $\rho = z\frac{d}{dz}$, where z is the complex coordinate on $\mathbb{C}$. Prove that $\operatorname{Lie}_\rho |z|^2 = |z|^2$.

b. Prove that $\operatorname{Lie}_\rho(\log|z|) = \mathsf{const}$.

Exercise 14.12. Let $z_1, ..., z_{n+1}$ be the complex coordinates on $\mathbb{C}^{n+1}$.

a. Prove that the vector fields $r := \sum_{i=1}^{n+1} z_i \frac{d}{dz_i}$ and $\bar r := \sum_{i=1}^{n+1} \bar z_i \frac{d}{d\bar z_i}$ are $\mathbb{C}^*$-invariant.

b. (!) Let $l(z_1, ..., z_n) = \sum |z_i|^2$. Prove that $\operatorname{Lie}_r(\log l) = 0$.

c. (!) Prove that $(d\log l) \lrcorner r = (d \log l) \lrcorner \bar r = 0$.

Hint. Use the previous exercise.

Exercise 14.13. Consider the tautological fibration $\mathbb{C}^{n+1}\backslash 0 \stackrel{\pi}{\longrightarrow} \mathbb{C}P^n$. We consider π as a quotient map, $\mathbb{C}P^n = (\mathbb{C}^{n+1}\backslash 0)/G$, where $G = \mathbb{C}^*$, and take $r = \sum_{i=1}^{n+1} z_i \frac{d}{dz_i}$ as above.

a. Prove that any $\mathbb{C}^*$-invariant form η such that $\eta \lrcorner r = \eta \lrcorner \bar{r} = 0$ is basic.

b. (!) Prove that $dd^c \log l$ is basic.

Exercise 14.14. Consider the tautological fibration $\mathbb{C}^{n+1}\backslash 0 \stackrel{\pi}{\longrightarrow} \mathbb{C}P^n$.

a. Prove that there exists a form $\omega \in \Lambda^{1,1}(\mathbb{C}P^1)$ such that $dd^c \log l = \pi^*(\omega)$.

b. Prove that this form is $U(n)$-invariant and has at least one positive eigenvalue.

c. (!) Prove that ω is a Kähler form.

Remark 14.1. This gives another definition of Fubini-Study form, clearly equivalent to the one we have seen in Handout 12.

Exercise 14.15 (!). Let $\mathbb{C}^n \subset \mathbb{C}P^n$ be an affine chart with affine coordinates $z_1, ..., z_n$. Prove that the Fubini-Study form on this chart is given by

$$\omega = \frac{\sum_{i=1}^n dz_i \wedge d\bar{z}_i}{1 + \sum_{i=1}^n |z_i|^2} - \frac{\sum_{i=1}^n \bar{z}_i dz_i}{1 + \sum_{i=1}^n |z_i|^2} \wedge \frac{\sum_{i=1}^n z_i d\bar{z}_i}{1 + \sum_{i=1}^n |z_i|^2}$$

Exercise 14.16 (!). Let $f_1, ..., f_n$ be holomorphic functions without common zeroes on a complex manifold. Prove that $\log(\sum_i |f_i|)$ is a plurisubharmonic function.

Exercise 14.17. Let L be a Hermitian holomorphic bundle on a complex manifold M, and $\operatorname{Tot} L$ its total space. Denote by $V \subset \operatorname{Tot} L$ the set of non-zero vectors in $\operatorname{Tot} L$. Clearly, V is equipped with a free action of $\mathbb{C}^*$, and $V/\mathbb{C}^* = M$. Denote by Σ the corresponding foliation. Consider the function $l:\ V \longrightarrow \mathbb{R}^{>0}$ mapping a vector $v \in L\big|_x$ to $|v|^2$.

a. (!) Prove that $dd^c \log l$ is a Σ-basic form on V.

b. (**) Prove that the corresponding (1,1)-form on M coincides with the curvature of the Chern connection on L.