Hodge theory

lecture 1: differential operators

NRU HSE, Moscow

Misha Verbitsky, January 24, 2018

William Vallance Douglas Hodge
17 June 1903 - 7 July 1975

Photograph by P. Halmos

Differential operators

Notation: Let M be a smooth manifold, TM its tangent bundle, $\Lambda^i M$ the bundle of differential i-forms, $C^\infty M$ the smooth functions. **The space of sections of a bundle B is denoted by B.**

DEFINITION: Let M be a manifold. The ring of **differential operators on the ring of functions on M** is a subalgebra of $\operatorname{End}_{\mathbb{R}}(C^\infty M, C^\infty M)$ is defined as follows. **Operator of order 0** is a $C^\infty M$-linear map, that is, a map $L_\alpha : f \mapsto \alpha f$, where $\alpha \in C^\infty M$ is a smooth function. **Operator of order 1** is a sum of a differentiation along a vector field and a $C^\infty M$-linear map. **Differential operator of order k** is a linear combination of products of k first order differential operators.

REMARK: In coordinates $x_1, ..., x_n$, differential operators can be expressed as sums of **differential monomials**:

$$D = f_0 + \sum_{i=1}^n f_i \frac{d}{dx_i} + \sum_{i,j=1}^n f_{ij} \frac{d^2}{dx_i dx_j} + \sum_{i,j,k=1}^n f_{ijk} \frac{d^3}{dx_i dx_j dx_k} + ...$$

Differential operators with coefficients in a trivial vector bundle

DEFINITION: Let E, F be trivial vector bundles on M, with basis $e_1, ..., e_n$ in E, $f_1, ..., f_m$ in F. **A differential operator from E to F** is a function mapping $\sum_{i=1}^n \alpha_i e_i$, where $\alpha_i \in C^\infty M$, to

$$D\left(\sum_{i=1}^n \alpha_i e_i\right) = \sum_{j=1}^m \sum_{i=1}^n D_{ij}(\alpha_i) f_j, \qquad (*)$$

where D_{ij} are differential operators on $C^\infty M$. One can think of D as a **$n \times m$-matrix with coefficients in differential operators on $C^\infty M$.**

Differential operators with coefficients in a vector bundle

DEFINITION: We say that a section b of a vector bundle B on M **has support in a set $K \subset M$** if b vanishes in an open set which contains $M \backslash K$. The smallest of all such K is called **support** of b.

DEFINITION: Let E, F be vector bundles on M. Let D be an operator mapping sections of E to sections of F. Suppose that for any open set $U \subset M$ such that E and F are trivial on U with bases $\{e_i\}$, $\{f_j\}$, and for any section $e = \sum_{i=1}^n \alpha_i e_i$ with support in U, the section $D(e)$ is expressed as in (*):

$$D\left(\sum_{i=1}^n \alpha_i e_i\right) = \sum_{j=1}^m \sum_{i=1}^n D_{ij}(\alpha_i) f_j.$$

Then D is called **a differential operator from E to F.**

EXAMPLE: Differential is a map $d: C^\infty M \longrightarrow \Lambda^1 M$ mapping a function to its differential. Prove that it is a first order differential operator.

EXAMPLE: A connection on a bundle B is an operator $\nabla: B \longrightarrow b \otimes \Lambda^1 M$ satisfying $\nabla(fb) = b \otimes df + f\nabla(b)$, where $f \longrightarrow df$ is de Rham differential. Prove that connection is a first order differential operator.

Local operators

DEFINITION: Let E, F be vector bundles on M. An operator D mapping sections of E to sections of F is called **local** if it maps any section with support in $K \subset M$ to a section with support in K.

REMARK: Differential operators are clearly local.

EXERCISE: (difficult)
Let M be a compact manifold, F, G – vector bundles. Prove that **any local operator from F to G is a differential operator.** Find a counterexample when M is non-compact.

Differential operators: algebraic definition

DEFINITION: (Grothendieck)
Let R be a commutative ring over a field k, and A, B R-modules. **Differential operator of order 0** from A to B is an R-linear map $\varphi \in \mathrm{Hom}_R(A,B)$. Differential operator of order $i > 0$ is defined inductively: $\alpha \in \mathrm{Diff}^i(A,B)$ if for any $r \in R$, the commutator $\alpha L_r - L_r\alpha$ belongs to $\mathrm{Diff}^{i-1}(A,B)$, where $L_r(x) = rx$.

DEFINITION: Given a vector bundle on a smooth manifold M, we may consider its space of sections as an $C^\infty M$-module. **Differential operators** $\mathrm{Diff}^i(F,G)$ on vector bundles F, G are defined as differential operators on the corresponding spaces of sections in the sense of the Grothendieck's definition. **Differential operator on M** is an element of $\mathrm{Diff}^i(M) := \mathrm{Diff}^i(C^\infty M, C^\infty M)$.

EXERCISE: Prove that this definition is equivalent to the usual one.

Discuss

0. Explain the format.

Discuss:

1. Language.

2. Time of the lectures and seminars.

The course's page:
`http://bogomolov-lab.ru/KURSY/Hodge-2018/`

REMINDER: de Rham algebra

DEFINITION: Let $\Lambda^* M$ denote the vector bundle with the fiber $\Lambda^* T_x^* M$ at $x \in M$ ($\Lambda^* T^* M$ is the Grassman algebra of the cotangent space $T_x^* M$). The sections of $\Lambda^i M$ are called **differential i-forms**. The algebraic operation "wedge product" defined on differential forms is $C^\infty M$-linear; the space $\Lambda^* M$ of all differential forms is called **the de Rham algebra**.

REMARK: $\Lambda^0 M = C^\infty M$.

THEOREM: There exists a unique operator $C^\infty M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \Lambda^2 M \xrightarrow{d} \Lambda^3 M \xrightarrow{d} ...$ satisfying the following properties

1. On functions, d is equal to the differential.
2. $d^2 = 0$
3. $d(\eta \wedge \xi) = d(\eta) \wedge \xi + (-1)^{\tilde\eta} \eta \wedge d(\xi)$, where $\tilde\eta = 0$ where $\eta \in \lambda^{2i} M$ is **an even form,** and $\eta \in \lambda^{2i+1} M$ is **odd.**

DEFINITION: The operator d is called **de Rham differential**.

EXERCISE: Prove it.

DEFINITION: A form η is called **closed** if $d\eta = 0$, **exact** if $\eta \in \operatorname{im} d$. The group $\frac{\ker d}{\operatorname{im} d}$ is called **de Rham cohomology** of M.

Graded algebras

DEFINITION: An algebra A is called **graded** if A is represented as $A = \bigoplus A^i$, where $i \in \mathbb{Z}$, and the product satisfies $A^i \cdot A^j \subset A^{i+j}$. Instead of $\bigoplus A^i$ one often writes A^*, where $*$ denotes all indices together. Some of the spaces A^i can be zero, but the ground field is always in A^0, so that it is non-empty.

EXAMPLE: The tensor algebra $T(V)$ and the polynomial algebra $\operatorname{Sym}^*(V)$ are obviously graded.

DEFINITION: Let $A^* = \oplus_{i\in\mathbb{Z}} A^i$ be a graded algebra over a field. It is called **graded commutative**, or **supercommutative**, if $ab = (-1)^{ij} ba$ for all $a \in A^i, b \in A^j$.

EXAMPLE: Grassmann algebra $\Lambda^* V$ is clearly supercommutative.

Graded derivations

DEFINITION: Let A^* be a graded commutative algebra, and $D: A^* \longrightarrow A^{*+i}$ be a map which shifts grading by i. It is called a **graded derivation**, if it satisfies the Leibniz rule: $D(ab) = D(a)b + (-1)^{ij} aD(b)$, for each $a \in A^j$.

DEFINITION: Let M be a smooth manifold, and $X \in TM$ a vector field. Consider an operation of **convolution with a vector field**

$$i_X: \Lambda^i M \longrightarrow \Lambda^{i-1} M,$$

mapping an i-form α to an $(i-1)$-form $v_1, ..., v_{i-1} \longrightarrow \alpha(X, v_1, ..., v_{i-1})$

EXERCISE: Prove that i_X is an odd derivation.

Supercommutator

DEFINITION: Let A^* be a graded vector space, and $E: A^* \longrightarrow A^{*+i}$, $F: A^* \longrightarrow A^{*+j}$ operators shifting the grading by i, j. Define **the supercommutator** $\{E, F\} := EF - (-1)^{ij} FE$.

DEFINITION: An endomorphism of a graded vector space which shifts grading by i is called **even** if i is even, and **odd** otherwise.

EXERCISE: Prove that the supercommutator satisfies **graded Jacobi identity,**

$$\{E, \{F, G\}\} = \{\{E, F\}, G\} + (-1)^{\tilde{E}\tilde{F}} \{F, \{E, G\}\}$$

where $\tilde{E}$ and $\tilde{F}$ are 0 if E, F are even, and 1 otherwise.

REMARK: There is a simple mnemonic rule which allows one to remember a superidentity, if you know the commutative analogue. **Each time when in commutative case two letters E, F are exchanged, in supercommutative case one needs to multiply by $(-1)^{\tilde{E}\tilde{F}}$.**

EXERCISE: **Prove that a supercommutator of superderivations is again a superderivation.**

Lie derivative

DEFINITION: Let B be a smooth manifold, and $v \in TM$ a vector field. An endomorphism $\operatorname{Lie}_v : \Lambda^* M \longrightarrow \Lambda^* M$, preserving the grading is called **a Lie derivative along v** if it satisfies the following conditions.

(1) On functions Lie_v is equal to a derivative along v. (2) $[\operatorname{Lie}_v, d] = 0$.

(3) Lie_v is a derivation of the de Rham algebra (that is, satisfies the Leibniz rule).

REMARK: The algebra $\Lambda^*(M)$ is generated by $C^\infty M = \Lambda^0(M)$ and $d(C^\infty M)$. The restriction $\operatorname{Lie}_v |_{C^\infty M}$ is determined by the first axiom. On $d(C^\infty M)$ is also determined because $\operatorname{Lie}_v(df) = d(\operatorname{Lie}_v f)$. **Therefore, Lie_v is uniquely defined by these axioms.**

EXERCISE: Prove the **anticommutator identity: $[d, \{d, E\}] = 0$ for each $E \in \operatorname{End}(\Lambda^* M)$.**

THEOREM: (Cartan's formula) Let i_v be a convolution with a vector field, $i_v(\eta) = \eta(v, \cdot, \cdot, ..., \cdot)$ **Then the anticommutator $\{d, i_v\}$ is equal to the Lie derivative along v.**

Proof: $\{d, \{d, i_v\}\} = 0$ by the lemma above. A supercommutator of two graded derivations is a graded derivation. Finally, $\{d, i_v\}$ acts on functions as $i_v(df) = \langle v, df \rangle$. ■

Connections

DEFINITION: Recall that **a connection** on a bundle B is an operator $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ satisfying $\nabla(fb) = b \otimes df + f\nabla(b)$, where $f \longrightarrow df$ is de Rham differential. When X is a vector field, we denote by $\nabla_X(b) \in B$ the term $\langle \nabla(b), X \rangle$.

REMARK: In local coordinates, connection on B is a sum of differential and a form $A \in \operatorname{End} B \otimes \Lambda^1 M$. Therefore, ∇_X is a derivation along X plus linear endomorphism. This implies that **each first order differential operator on B is expressed as a linear combination of the compositions of covariant derivatives ∇_X and linear maps.**

This follows from the definition of the first order differential operator: **by definition, it is a linear combination of partial derivatives combined with a linear maps.**

Connection and a tensor product

REMARK: A connection ∇ on B gives a connection $B^* \xrightarrow{\nabla^*} \Lambda^1 M \otimes B^*$ on the dual bundle, by the formula

$$d(\langle b, \beta\rangle) = \langle \nabla b, \beta\rangle + \langle b, \nabla^* \beta\rangle$$

These connections are usually denoted **by the same letter ∇.**

REMARK: For any tensor bundle $\mathcal{B}_1 := B^* \otimes B^* \otimes ... \otimes B^* \otimes B \otimes B \otimes ... \otimes B$ **a connection on B defines a connection on $\mathcal{B}_1$** using the Leibniz formula:

$$\nabla(b_1 \otimes b_2) = \nabla(b_1) \otimes b_2 + b_1 \otimes \nabla(b_2).$$

Adjoint connection

DEFINITION: Given a connection ∇ on a vector bundle B equipped with a scalar product $(\cdot,\cdot)$, define ∇^* by the formula

$$d(b,b') = (\nabla(b),b') + (b,\nabla^*(b')). \quad (**)$$

Here, b,b' are sections of B, $d(b,b')$ is a differential of a function, and $(\nabla(b),b')$ is the 1-form obtained from the bilinear pairing $B \otimes (B \otimes \Lambda^1 M) \longrightarrow \Lambda^1 M$.

CLAIM: The map $\nabla^* : B \longrightarrow B \otimes \Lambda^1 M$ is well defined by (**). Moreover, it is also a connection.

Proof: The first statement is clear, because any linear map $B \longrightarrow \Lambda^1 M$ can be represented by $b \longrightarrow (b,A)$ for some $A \in B \otimes \Lambda^1 M$. To check the second statement, we take $f \in C^\infty M$, and write

$$(b,b')df + fd(b,b') = d(b,fb') = f(\nabla(b),b') + (b,\nabla^*(fb')).(**)$$

which gives $\Big(b,\nabla^*(fb') - f\nabla^*(b')\Big) = (b,b')df$, hence $\nabla^*(fb') - f\nabla^*(b') = b' \otimes df$.
■

DEFINITION: The connection ∇^* is called **adjoint connection** to ∇. Relation $\nabla = \nabla^*$ happens precisely when ∇ preserves the metric tensor, considered as a section of $B^* \otimes B^*$, and in this case ∇ is called **an orthogonal connection**.

Adjoint connection and L^2-product

DEFINITION: Fix a volume form Vol on a manifold M Consider a $C^\infty M$-linear scalar product on a vector bundle B. Then **the space of sections of B is also equipped with a scalar product:** $(b,b')_{L^2} = \int_M (b,b')\,\mathrm{Vol}$. It is called **the standard L^2-scalar product on the space of sections**.

LEMMA: (integration by parts)
Let B be a bundle on M with scalar product and connection ∇, and $b, b' \in B$ its sections. **Then, for any vector fields $X \in TM$, one has**

$$\int_M (\nabla_X b, b') + \int_M (b, \nabla^*_X b') = \int_M (b,b')\,\mathrm{Lie}_X \mathrm{Vol} \quad (***)$$

Proof: By definition, one has

$$\int_M ((\nabla_X)^*(b), b')\,\mathrm{Vol} = \int_M (b, \nabla_X b')\,\mathrm{Vol} = -\int_M (\nabla^*_X b, b')\,\mathrm{Vol} - \int_M \mathrm{Lie}_X(b,b')\,\mathrm{Vol},$$

where $\mathrm{Lie}_X(b,b')$ is differential of the function (b,b') along $X \in TM$. However, for any top form η, one has $\mathrm{Lie}_X(\eta) = d(i_x\eta)$ by Cartan's formula, giving $\int_M \mathrm{Lie}_X(\eta) = 0$, hence

$$0 = \int_M \mathrm{Lie}_X((b,b')\,\mathrm{Vol}) = \int_M \mathrm{Lie}_X(b,b')\,\mathrm{Vol} + \int_M (b,b')\,\mathrm{Lie}_X \mathrm{Vol},$$

giving the last term in (***). ■

Adjoint operators

REMARK: Operators $A: F \longrightarrow G$ and $A^*: G \longrightarrow F$ on spaces with a scalar product are called **orthogonal adjoint**, or **adjoint**, if $(A(f), g) = (f, A^*(g))$ for each $f \in F$, $g \in G$.

CLAIM: An orthogonal adjoint D^* to a differential operator D **is a differential operator again.**

Proof. Step 1: This is clear for $C^\infty M$-linear operators (just take the pointwise adjoint map). **If we prove it for first order operators, we are done,** because $(XY)^* = Y^*X^*$.

Step 2: First order operators are expressed as linear combination of linear maps and derivatives $\nabla_X: F \longrightarrow F$ combined with linear maps. Therefore, **it would suffice to show that $(\nabla_X)^*$ is a differential operator.**

Step 3: The map $(\nabla_X)^*$ is a differential operator: $(\nabla_X)^*(b) = -\nabla_X^* - \frac{\operatorname{Lie}_X(\operatorname{Vol})}{\operatorname{Vol}} b$, because

$$\int_M ((\nabla_X)^*(b), b')\operatorname{Vol} = -\int_M (\nabla_X^* b, b')\operatorname{Vol} - \int (b, b')\operatorname{Lie}_X(\operatorname{Vol})$$

by "integration by parts", as shown above. ∎

Laplacian on differential forms

DEFINITION: Let V be a vector space. **A metric g on V induces a natural metric on each of its tensor spaces:** $g(x_1 \otimes x_2 \otimes ... \otimes x_k, x'_1 \otimes x'_2 \otimes ... \otimes x'_k) = g(x_1, x'_1)g(x_2, x'_2)...g(x_k, x'_k)$.

This gives a natural positive definite scalar product on differential forms over a Riemannian manifold (M, g): $g(\alpha, \beta) := \int_M g(\alpha, \beta) \operatorname{Vol}_M$

DEFINITION: Let M be a Riemannian manifold. **Laplacian on differential forms** is $\Delta := dd^* + d^*d$.

REMARK: Laplacian is self-adjoint and positive definite: $(\Delta x, x) = (dx, dx) + (d^*x, d^*x)$. Also, Δ commutes with d and d^*.

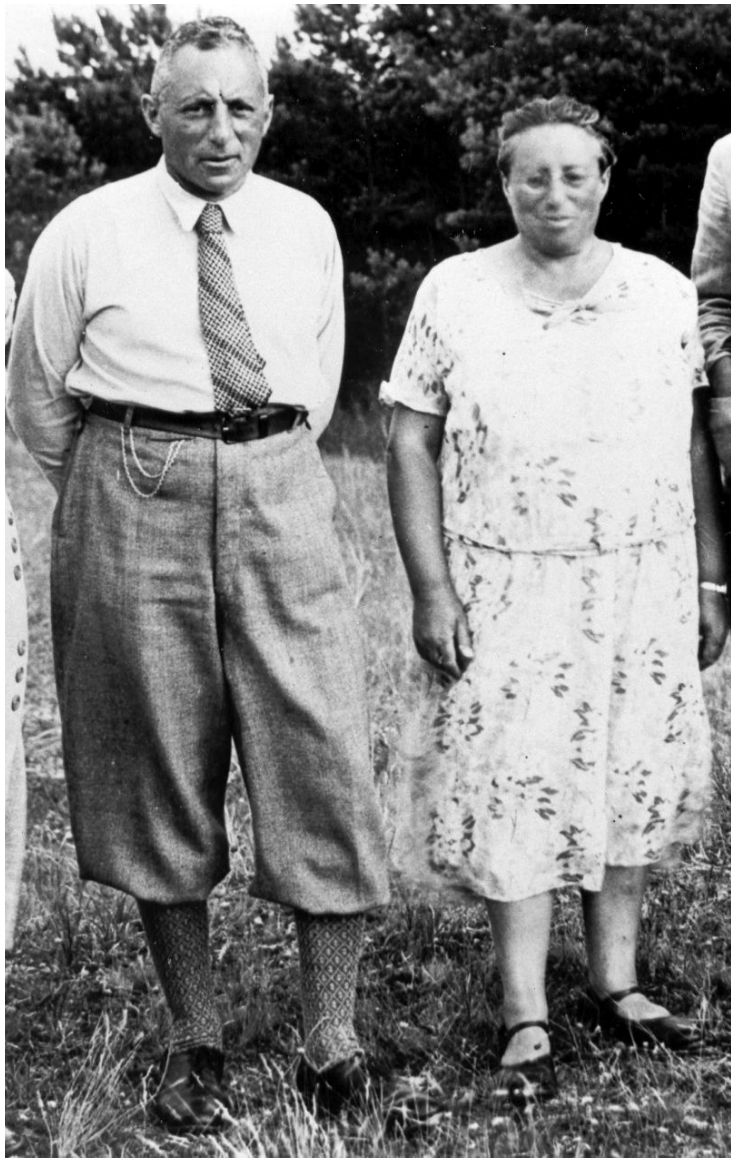
THEOREM: (The main theorem of Hodge theory)
There is an orthonormal basis in the Hilbert space $L^2(\Lambda^*(M))$ consisting of eigenvectors of Δ.

THEOREM: ("Elliptic regularity for Δ") Let $\alpha \in L^2(\Lambda^k(M))$ be an eigenvector of Δ. **Then α is a smooth k-form.**

These two theorems will be proven later.

Fritz Alexander Ernst Noether

(October 7, 1884 - September 10, 1941)



Emmy Noether und Fritz Noether, 1933

De Rham cohomology

DEFINITION: The space $H^i(M) := \frac{\ker d\big|_{\Lambda^i M}}{d(\Lambda^{i-1}M)}$ is called **the de Rham cohomology of M**.

DEFINITION: A form α is called **harmonic** if $\Delta(\alpha) = 0$.

REMARK: Let α be a harmonic form. **Then** $(\Delta x, x) = (dx, dx) + (d^* x, d^* x)$, hence $\alpha \in \ker d \cap \ker d^*$.

REMARK: The projection $\mathcal{H}^i(M) \longrightarrow H^i(M)$ from harmonic forms to cohomology is injective. Indeed, a form α lies in the kernel of such projection if $\alpha = d\beta$, but then $(\alpha, \alpha) = (\alpha, d\beta) = (d^*\alpha, \beta) = 0$.

THEOREM: The natural map $\mathcal{H}^i(M) \longrightarrow H^i(M)$ is an isomorphism (see the next page).

REMARK: Poincare duality immediately follows from this theorem.

Hodge theory and the cohomology

THEOREM: The natural map $\mathcal{H}^i(M) \longrightarrow H^i(M)$ is an isomorphism.

Proof. Step 1: Since $d^2 = 0$ and $(d^*)^2 = 0$, one has $\{d, \Delta\} = 0$. This means that **Δ commutes with the de Rham differential.**

Step 2: Consider the eigenspace decomposition $\Lambda^*(M) \widehat{\cong} \bigoplus_\alpha \mathcal{H}^*_\alpha(M)$, where α runs through all eigenvalues of Δ, and $\mathcal{H}^*_\alpha(M)$ is the corresponding eigenspace. **For each α, de Rham differential defines a complex**

$$\mathcal{H}^0_\alpha(M) \xrightarrow{d} \mathcal{H}^1_\alpha(M) \xrightarrow{d} \mathcal{H}^2_\alpha(M) \xrightarrow{d} \dots$$

Step 3: On $\mathcal{H}^*_\alpha(M)$, one has $dd^* + d^*d = \alpha$. When $\alpha \neq 0$, and η closed, this implies $dd^*(\eta) + d^*d(\eta) = dd^*\eta = \alpha\eta$, hence $\eta = d\xi$, with $\xi := \alpha^{-1}d^*\eta$. This implies that **the complexes $(\mathcal{H}^*_\alpha(M), d)$ don't contribute to cohomology.**

Step 4: We have proven that

$$H^*(\Lambda^* M, d) = \bigoplus_\alpha H^*(\mathcal{H}^*_\alpha(M), d) = H^*(\mathcal{H}^*_0(M), d) = \mathcal{H}^*(M).$$

■