Hodge theory

lecture 2: elliptic operators

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Differential operators (reminder)

Notation: Let M be a smooth manifold, TM its tangent bundle, $\Lambda^i M$ the bundle of differential i-forms, $C^\infty M$ the smooth functions. **The space of sections of a bundle B is denoted by B.**

DEFINITION: Let M be a manifold. The ring of **differential operators on the ring of functions on M** is a subalgebra of $\operatorname{End}_{\mathbb{R}}(C^\infty M, C^\infty M)$ is defined as follows. **Operator of order 0** is a $C^\infty M$-linear map, that is, a map $L_\alpha : f \mapsto \alpha f$, where $\alpha \in C^\infty M$ is a smooth function. **Operator of order 1** is a sum of a differentiation along a vector field and a $C^\infty M$-linear map. **Differential operator of order k** is a linear combination of products of k first order differential operators.

REMARK: In coordinates $x_1, ..., x_n$, differential operators can be expressed as sums of **differential monomials**:

$$D = f_0 + \sum_{i=1}^n f_i \frac{\partial}{\partial x_i} + \sum_{i,j=1}^n f_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i,j,k=1}^n f_{ijk} \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} + ...$$

Differential operators with coefficients in a vector bundle (reminder)

DEFINITION: Let E, F be trivial vector bundles on M, with basis $e_1, ..., e_n$ in E, $f_1, ..., f_m$ in F. **A differential operator from E to F** is a function mapping $\sum_{i=1}^n \alpha_i e_i$, where $\alpha_i \in C^\infty M$, to

$$D\left(\sum_{i=1}^n \alpha_i e_i\right) = \sum_{j=1}^m \sum_{i=1}^n D_{ij}(\alpha_i) f_j, \qquad (*)$$

where D_{ij} are differential operators on $C^\infty M$. One can think of D as a **$n \times m$-matrix with coefficients in differential operators on $C^\infty M$.**

DEFINITION: We say that a section b of a vector bundle B on M **has support in a set $K \subset M$** if b vanishes in an open set which contains $M \backslash K$. The smallest of all such K is called **support** of b.

DEFINITION: Let E, F be vector bundles on M. Let D be an operator mapping sections of E to sections of F. Suppose that for any open set $U \subset M$ such that E and F are trivial on U with bases $\{e_i\}$, $\{f_j\}$, and for any $e = \sum_{i=1}^n \alpha_i e_i$ with support in U, the section $D(e)$ is expressed as in (*):

$$D\left(\sum_{i=1}^n \alpha_i e_i\right) = \sum_{j=1}^m \sum_{i=1}^n D_{ij}(\alpha_i) f_j.$$

Then D is called **a differential operator from E to F.**

Associated graded rings

REMARK: Algebra is an associative ring over a field. Rings in this lecture **are not necessarily commutative, but always associative.**

DEFINITION: Let R be an associative ring. **Filtration** on R is a collection of subspaces $R_0 \subset R_1 \subset R_2 \subset ...$ such that $R_i R_j \subset R_{i+j}$.

REMARK: Let $x \in R_k, y \in R_l$. Then the product xy modulo R_{k+l-1} depends only on the class of x modulo R_{k-1}. Indeed, $R_{k-1}R_l \subset R_{k+l-1}$. **This defines the product map** $(R_k/R_{k-1}) \otimes (R_l/R_{l-1}) \longrightarrow R_{k+l}/R_{k+l-1}$**. We obtained the associative product structure on the space** $\bigoplus_{i=0}^{\infty} R_i/R_{i-1}$**.**

DEFINITION: Let $R_0 \subset R_1 \subset R_2 \subset ...$ be a filtered ring. The ring $\bigoplus_{i=0}^{\infty} R_i/R_{i-1}$ is called **the associated graded ring** of this filtration.

EXAMPLE: Consider the filtration $\operatorname{Diff}^0(M) \subset \operatorname{Diff}^1(M) \subset \operatorname{Diff}^2(M) \subset ...$ on the ring of differential operators. The associated graded ring is called **the ring of symbols of differential operators**.

Order of zeroes of a function

DEFINITION: Let $m \in M$, and $x_1, ..., x_n$ a coordinate system around m, with $x_1(m) = x_2(m) = ... = x_n(m) = 0$. We say that **a function f has zero of order $\geqslant k$ at m** if $\frac{\partial^l f}{\partial x_{i_1} \partial x_{i_2} ... \partial x_{i_l}}(m) = 0$ for any $l < k$.

CLAIM: Let $\mathfrak{m} \subset C^\infty M$ be the ideal of all functions vanishing in $m \in M$. **Then f has zero of order $\geqslant k$ at m if and only if $f \in \mathfrak{m}^k$.**

Proof. Step 1: Let $f \in \mathfrak{m}^k$. Then $\frac{\partial f}{\partial x_i} \in \mathfrak{m}^{k-1}$ by Leibniz rule. **Hence f has zero of order $\geqslant k$ at m.**

Step 2: The function f has zero of order $\geqslant k$ at m if and only if $\frac{\partial f}{\partial x_i}$ has zero of order $\geqslant k - 1$ at m.

Step 3: If f has zero of order $\geqslant 1$ at m, this means that $f \in \mathfrak{m}$ by definition. Using induction in k and step 2, we obtain that **f has zero of order $\geqslant k$ at $m \Leftrightarrow \frac{\partial f}{\partial x_i} \in \mathfrak{m}^{k-1} \Leftrightarrow f \in \mathfrak{m}^k$.** ■

COROLLARY: Consider a function f with the Taylor series decomposition $f = \sum P_i$ in m, where $P_i \in \mathbb{R}[x_1, ..., x_n]$ a homogeneous polynomial of degree i. **Then f has zero of order $\geqslant k$ in m if and only if $P_0 = P_1 = ... = P_{k-1} = 0$.**
■

Differential operators: algebraic definition

DEFINITION: (Grothendieck) Let R be a commutative ring over a field k. Given $a \in R$, consider the map $L_a: R \longrightarrow R$ mapping x to ax. Define $\operatorname{Diff}^k(R) \subset \operatorname{Hom}_k(R,R)$ inductively as follows. The $\operatorname{Diff}^0(R)$ is the space of all R-linear maps from R to R, that is, the space of all L_a, $a \in R$. The space $\operatorname{Diff}^k(R)$, $k > 0$ is

$$\operatorname{Diff}^k(R) := \{D \in \operatorname{Hom}_k(R,R) \mid [L_a, D] \in \operatorname{Diff}^{k-1}(R) \forall a \in R.\}$$

EXERCISE: Prove that $\operatorname{Diff}^k(C^\infty M)$ **in this sense is the same as the space of differential operators defined in the standard way.**

CLAIM: For any $D \in \operatorname{Diff}^{k-1}(C^\infty M)$, and any $a \in C^\infty M$, **one has** $[L_a, D] \in \operatorname{Diff}^{k-1}(C^\infty M)$.

Proof: Indeed, for a vector field $X \in TM$, one has $[L_a, \operatorname{Lie}_X] = L_{\operatorname{Lie}_X(a)}$, which means that $[L_a, \operatorname{Diff}^1(C^\infty M)] \subset \operatorname{Diff}^0(C^\infty M)$. Now, if we take a commutator of L_a and a product of k elements from $\operatorname{Diff}^1(C^\infty M)$, we obtain a linear combination of products of $k-1$ elements of $\operatorname{Diff}^1(C^\infty M)$, by Leibniz formula:

$$[L_a, D_1 D_2 ... D_n] = [[L_a, D_1] D_2 ... D_n + D_1 [L_a, D_2] D_3 ... D_n + ... + D_1 D_2 ... D_{n-1} [L_a, D_n].$$

■

Differential operators and homogeneous polynomials

LEMMA: Let $x_1, ..., x_n$ be coordinates on $\mathbb{R}^n$, and $D \in \operatorname{Diff}^k(\mathbb{R}^n)$ a differential operator vanishing on all homogeneous polynomials $P \in \mathbb{R}[x_1, ..., x_n]$ of degree k. **Then $D = 0$.**

Proof. Step 1: We prove lemma by induction. For $k = 0$ it is clear. Let $L_{x_i}: C^\infty M \longrightarrow C^\infty M$ be the multiplication by x_i. Then $[L_{x_i}, D]$ is a differential operator of order $k-1$ vanishing on all homogeneous polynomials of degree $\leqslant k-1$. Using induction on k, we obtain that $[L_{x_i}, D] = 0$. Then D is $\mathbb{R}[x_1, ..., x_n]$-linear. Since D vanishes on polynomials of degree k and is $\mathbb{R}[x_1, ..., x_n]$-linear, it vanishes on all polynomials.

Step 2: Let $f \in C^\infty \mathbb{R}^n$ and $x \in \mathbb{R}^n$ be a point. Using Hadamar's lemma, we obtain that $f = P + f_0$, where $P \in \mathbb{R}[x_1, ..., x_n]$ is a polynomial of degree k and f_0 has zero of order $\geqslant k+1$ in x. Then $D(f)(x) = D(P) + D(f_0) = 0$ (the first summand vanishes because D is $\mathbb{R}[x_1, ..., x_n]$-linear, and the second because D is of order k and f_0 has zero of order $\leqslant k+1$ in x). ■

The ring of symbols

THEOREM: Consider the filtration $\operatorname{Diff}^0(M) \subset \operatorname{Diff}^1(M) \subset \operatorname{Diff}^2(M) \subset ...$ on the ring of differential operators. Then **its associated graded ring is isomorphic to the ring $\bigoplus_i \operatorname{Sym}^i(TM)$.**

Proof. Step 1: Let f be a function with zero of order $\geqslant k$ in z, and $\mathfrak{m}$ its maximal ideal. Then $\operatorname{Diff}^{k-1}(f) = 0$. This gives a bilinear pairing

$$(\operatorname{Diff}^k(M)/\operatorname{Diff}^{k-1}(M)) \times (\mathfrak{m}^k)/(\mathfrak{m}^{k-1}) \longrightarrow \mathbb{R}$$

mapping $D \otimes f$ to $Df(0)$. Since $(\mathfrak{m}^k)/(\mathfrak{m}^{k-1}) = \operatorname{Sym}^i(T_z M)^*$, this pairing, applied to all $z \in M$ gives a natural map $\operatorname{Diff}^k(M)/\operatorname{Diff}^{k-1}(M) \xrightarrow{\sigma} \operatorname{Sym}^i(TM)$. **It remains to prove that σ is an isomorphism.** Since this pairing is local, it suffices to prove that it is an isomorphism for $M = \mathbb{R}^n$.

Step 2: Let $x_1, ..., x_n$ be coordinates in $M = \mathbb{R}^n$, and $D \in \operatorname{Diff}^k(M)$. Then $\operatorname{Sym}^*(TM) = C^\infty M[\frac{d}{dx_1}, \frac{d}{dx_2}, ..., \frac{d}{dx_n}]$. Consider a homogeneous differential monomial $D = f\frac{\partial^k}{\partial x_{i_1} ... \partial x_{i_k}}$. Then $\sigma(D) = f\frac{d}{dx_{i_1}}\frac{d}{dx_{i_2}} ... \frac{d}{dx_{i_k}}$. **Therefore, σ is surjective.**

Step 3: Let $D \in \operatorname{Diff}^k(\mathbb{R}^n)$ be a differential operator and $\underline{D}$ its class in $\operatorname{Diff}^k(\mathbb{R}^n)/\operatorname{Diff}^{k-1}(\mathbb{R}^n)$ such that $\sigma(\underline{D}) = 0$. Since σ is evaluation on polynomials, D vanishes on all homogeneous polynomials of degree k. By Lemma 1 above, $D = 0$. ■

Symbols

THEOREM: Consider the filtration $\operatorname{Diff}^0(M) \subset \operatorname{Diff}^1(M) \subset \operatorname{Diff}^2(M) \subset ....$ Then **its associated graded ring is isomorphic to $\bigoplus_i \operatorname{Sym}^i(TM)$,** identified with the ring if fiberwise polynomial functions on T^*M.

COROLLARY: Let F, G be vector bundles, and $\operatorname{Diff}^0(F,G) \subset \operatorname{Diff}^1(F,G) \subset \operatorname{Diff}^2(F,G)$ the corresponding spaces of differential operators. **Then**

$$\operatorname{Diff}^i(F,G)/\operatorname{Diff}^{i-1}(F,G) = \operatorname{Sym}^i(TM) \otimes \operatorname{Hom}(F,G),$$

where Sym^i denotes the symmetric power (symmetric part of the tensor power).

DEFINITION: Let F, G be vector bundles, and $D \in \operatorname{Diff}^i(F,G)$ a differential operator. Consider its class in $\operatorname{Diff}^i(F,G)/\operatorname{Diff}^{i-1}(F,G)$ as a $\operatorname{Hom}(F,G)$-valued function on $T^*(M)$ (polynomial of order i on each cotangent space). This function is called **the symbol** of D.

EXERCISE: Let $D: \ B \longrightarrow B \otimes \Lambda^1 M$ be a first order differential operator. **Prove that D is a connection if and only if its symbol is equal to the identity operator** $\operatorname{Id} \in \operatorname{Hom}(\Lambda^1 M \otimes (\operatorname{Hom}(B, B \otimes \Lambda^1 M))$

EXERCISE: Prove that the symbol of the Laplacian operator $\Delta: \ \Lambda^* M \longrightarrow \Lambda^* M$ on a Riemannian manifold M at $\xi \in T^*M$ **is equal to $|\xi|^2 \operatorname{Id}_{\Lambda^* M}$.**

Elliptic operators

DEFINITION: Let F, G be vector bundles of the same rank. A differential operator $D: F \longrightarrow G$ is called **elliptic** if its symbol $\sigma(D) \in \operatorname{Hom}(F,G) \otimes \operatorname{Sym}^i(TM)$ is invertible at each non-zero $\xi \in T^*M$.

EXAMPLE: Consider an operator of second order on $C^\infty(\mathbb{R}^n)$,

$$D = f_0 + \sum_{i=1}^n f_i \frac{\partial}{\partial x_i} + \sum_{i,j=1}^n f_{ij} \frac{\partial^2}{\partial x_i \partial x_j}.$$

Then the symbol of D is $f_{ij}\frac{\partial^2}{\partial x_i \partial x_j}$; **it is elliptic if and only if the symmetric form $f_{ij} dx_i dx_j$ is positive or negative definite everywhere in $\mathbb{R}^n$.**

EXERCISE: Prove that symbol of D^* is a $\operatorname{Hom}(F,G)$-valued function on $T^*(M)$ which is Hermitian adjoint to $\operatorname{symb}(D)$.

Elliptic operators: main properties

The rest of the slides today are introduction to the main results about elliptic operators; complete proofs for some of them will be given later.

THEOREM: (Elliptic regularity)
Let D be an elliptic operator with smooth coefficients on a manifold (not necessarily compact), and $f \in \ker D$. **Then D is smooth, and real analytic if coefficients of D were real analytic.**

THEOREM: Let D be an elliptic operator with smooth coefficients on a compact manifold. **Then its kernel is finite-dimensional.** If $D: F \longrightarrow F$ is self-adjoint, then **D can be diagonalized in an appropriate orthonormal basis in the space of sections of F, and its eigenvalues are discrete.**

Elliptic operators of second order

Let $D = f_0 + \sum_{i=1}^n f_i \frac{\partial}{\partial x_i} + \sum_{i,j=1}^n f_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$ be an elliptic operator of second order. For second order operators, **we always assume that the symbol $f_{ij} dx_i dx_j$ is positive definite.**

THEOREM: (E. Hopf's maximum principle)
Let D be a second order elliptic operator on a manifold (not necessarily compact) such that $D(\text{const}) = 0$, and f a solution of an equation $D(f) = 0$. Assume that f has a local maximum. **Then $f = \text{const}$.**

THEOREM: (Harnack inequality)
Let D be a second order elliptic operator on a manifold M (not necessarily compact), and $\Omega \subset M$ an open subset with compact closure. For any $f \in \mathbb{C}^\infty(\Omega)$, denote by $\operatorname{Var}_\Omega(f)$ the number $\sup_\Omega(f) - \inf_\Omega(f)$. **Then there exists a constant C, depending only on D, M and Ω, such that for any $f \in \ker D$, one has $\operatorname{Var}_\Omega(f) < C$.**

COROLLARY: Any pointwise converging sequence of functions $f \in \ker D$ converges uniformly.

Proof: Indeed, by Harnack's inequality, the solutions of $D(f) = 0$ are uniformly continuous, hence the pointwise convergence implies uniform convergence. ■

Eberhard Hopf (1902-1983)

Fredholm operators

DEFINITION: Let F be a vector bundle on a compact manifold. The **L^2_p-topology** on the space of sections of F is a topology defined by a quadratic form $|f|^2 = \sum_{i=0}^p \int_M |\nabla^i f|^2$, for some connection and scalar product on F and $\Lambda^1 M$.

EXERCISE: Prove that this topology is independent from the choice of a connection and a metric.

DEFINITION: A continuous operator $\psi: A \longrightarrow B$ on topological vector spaces is called **Fredholm** if its kernel is finite-dimensional, and its image is closed, and has finite codimension.

THEOREM: Let $D: F \longrightarrow G$ be an elliptic operator of order d. Clearly, D defines a continuous map $L^2_p(F) \longrightarrow L^2_{p-d}(G)$. **Then this map is Fredholm.**

REMARK: This difficult theorem is a foundation of Hodge theory (and many other things).

Index of an elliptic operator

REMARK: Let $D: F \longrightarrow G$ be an elliptic operator of order d, and $L^2_p(F) \longrightarrow L^2_{p-d}(G)$ the corresponding maps on L^p-spaces.

DEFINITION: Index of a Fredholm operator D is the number $\dim\ker D - \dim\operatorname{coker} D = \dim\ker D - \dim\ker D^*$

REMARK: Index of an elliptic operator $D: L^2_p(F) \longrightarrow L^2_{p-d}(G)$ a priori depends on p, however, by elliptic regularity, **all elements of $\ker D$ and $\ker D^*$ are smooth,** hence $\dim\ker D\big|_{L^2_p(F)}$ is independent from p.

EXERCISE: Let F_t be a continuous family of Fredholm operators. **Prove that the index of F_t is constant in t.**

COROLLARY: Let D_t be a continuous family of elliptic operators of order k. **Then $\operatorname{ind} D_0 = \operatorname{ind} D_1$.**

COROLLARY: Index of an elliptic operator **is determined by its symbol.**

Proof: For any elliptic operators D_0, D_1 with the same symbol, the operator $D_t := tD_1 + (1-t)D_0$ has the same symbol, hence D_0 can be deformed to D_0 continuously and **gives a continuous family of Fredholm operators.** ■

Atiyah-Singer index theorem

REMARK: The index of an elliptic operator is clearly constant under continuous change of its symbol. Therefore, **index depends only on the homotopy class of its symbol**, which can be considered as a non-degenerate section of $\operatorname{Hom}(F,G)$ over $T^*M\backslash 0$. Homotopy classes of such sections are described explicitly in terms of characteristic classes of F, G and the topological K-theory of M. **Atiyah-Singer index formula** expresses the index of an elliptic operator as a polynomial function of these topological invariants.

EXAMPLE: Let M be a compact manifold, and $D: C^\infty M \longrightarrow C^\infty M$ elliptic operator of second order. **Then** $\operatorname{ind} D = 0$**.**

Proof: Locally, we can write $D = f_0 + \sum_{i=1}^n f_i \frac{\partial}{\partial x_i} + \sum_{i,j=1}^n f_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$. The Laplacian Δ associated with the metric tensor f_{ij} has the same symbol, hence it suffices to prove $\operatorname{ind} \Delta = 0$. However, Δ is self-adjoint, hence $\dim \ker \Delta = \dim \operatorname{coker} \Delta$. ■

COROLLARY: For any second order elliptic operator D on $C^\infty M$ with $D(\text{const}) = 0$, **one has** $\dim \operatorname{coker} D = 1$**.**

Proof: By the strong maximum principle, all functions $\ker D$ are constant, hence $\dim \ker D = 1$. **Then** $\dim \operatorname{coker} D = 1$ **by the index theorem.** ■