Hodge theory

lecture 3: compact operators

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Hilbert spaces

DEFINITION: Hilbert space is a complete, infinite-dimensional Hermitian space which is second countable (that is, has a countable dense set).

DEFINITION: Orthonormal basis in a Hilbert space H is a set of pairwise orthogonal vectors $\{x_\alpha\}$ which satisfy $|x_\alpha| = 1$, and such that H is the closure of the subspace generated by the set $\{x_\alpha\}$.

THEOREM: Any Hilbert space has a basis, and all such bases are countable.

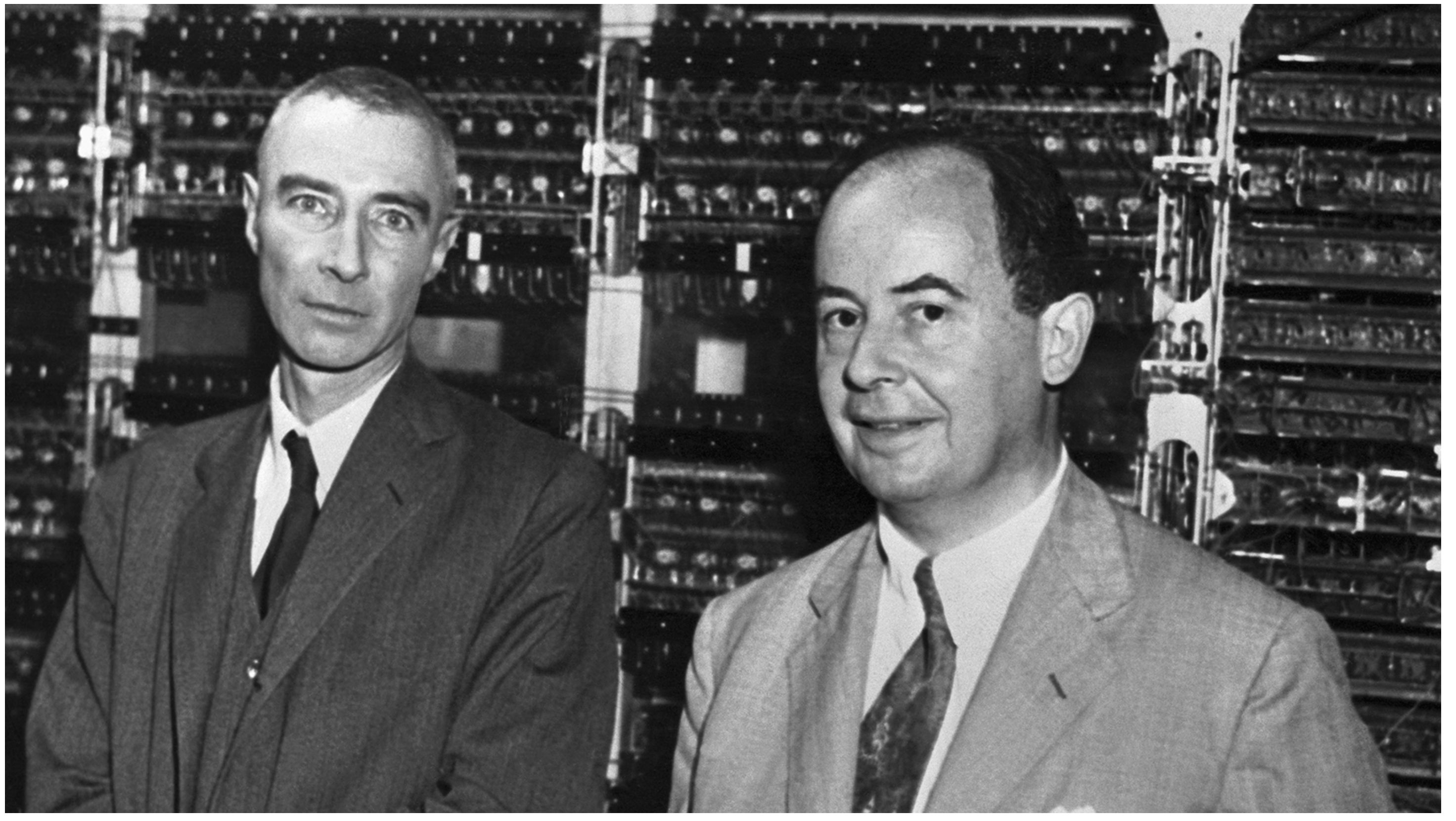
Proof: A basis is found using Zorn lemma. If it's not countable, open balls with centers in x_α and radius $\varepsilon < 2^{-1/2}$ don't intersect, which means that the second countability axiom is not satisfied. ■

THEOREM: All Hilbert spaces are isometric.

Proof: Each Hilbert space has a countable orthonormal basis. ■

REMARK: The concept of "Hilbert space" and this theorem is due to John von Neumann (1927); he's also responsible for the spectral theorem for compact operators.

John von Neumann, 1903-1957



Dr. John Von Neumann (right) stands with Dr. J. Robert Oppenheimer in front of an early computer.

...Although Max insisted von Neumann attend school at the grade level appropriate to his age, he agreed to hire private tutors to give him advanced instruction in those areas in which he had displayed an aptitude. At the age of 15, he began to study advanced calculus under the renowned analyst Gábor Szegő. On their first meeting, Szegő was so astounded with the boy's mathematical talent that he was brought to tears...

Real Hilbert spaces

DEFINITION: A Euclidean space is a vector space over $\mathbb{R}$ equipped with a positive definite scalar product g.

DEFINITION: **Real Hilbert space** is a complete, infinite-dimensional Euclidean space which is second countable (that is, has a countable dense set).

DEFINITION: **Orthonormal basis** in a Hilbert space H is a set of pairwise orthogonal vectors $\{x_\alpha\}$ which satisfy $|x_\alpha| = 1$, and such that H is the closure of the subspace generated by the set $\{x_\alpha\}$.

THEOREM: **Any real Hilbert space has a basis, and all such bases are countable.**

Proof: A basis is found using Zorn lemma. If it's not countable, open balls with centers in x_α and radius $\varepsilon < 2^{-1/2}$ don't intersect, which means that the second countability axiom is not satisfied. ■

THEOREM: **All real Hilbert spaces are isometric**.

Proof: Each Hilbert space has a countable orthonormal basis. ■

Adjoint operators (reminder)

CLAIM: Let V be a Hilbert space (real or complex), g a scalar product on V, and $A \in \operatorname{End}(V)$. **Then there exists a unique operator $A^* \in \operatorname{End}(V)$ such that $g(A(x), y) = g(x, A^*(y))$ for all $x, y \in V$.**

Proof: Let $x_1, ..., x_n, ...$ be an orthonormal basis in V, $A = (a_{ij})$ the matrix of A. Then $g(A(x_i), x_j) = a_{ij}$ and $g(x_i, A^*(x_j)) = \overline{a}_{ij}$. This gives the existence. Uniqueness is clear, because if $g(x, (A_1^* - A_2^*)(y)) = 0$ for all x, y, we have $A_1^* - A_2^* = 0$ **(prove it).** ■

DEFINITION: In this situation, the operator A^* is called **orthogonal adjoint** or **Hermitian adjoint** to A. In orthonormal basis, **this operator is represented by the transposed matrix A^t** for real Hilbert spaces and by $\overline{A}^t$ for complex Hilbert spaces.

Self-adjoint operators

DEFINITION: Let V be a vector space and $g \in \operatorname{Sym}^2 V$ a scalar product. An operator $A: V \longrightarrow V$ is called **self-adjoint** if $A = A^*$.

REMARK: If we work in a real vector space orthonormal basis, a self-adjoint operator is given by a matrix that satisfies $A = A^t$, that is, **symmetric**. The self-adjoint operators are often called **symmetric operators**. For complex spaces, A is Hermitian if and only if $A^t = \overline{A}$; such operators are called **Hermitian**.

Assume that V is finitely-dimensional.

CLAIM: Let A be a (real) self-adjoint operator on (V, g), and $g_A(x, y) := g(A(x), y)$. **Then g_A is a bilinear symmetric form on V.** Moreover, **the map $A \mapsto g_A$ gives a bijective correspondence between self-adjoint operators and bilinear symmetric forms on V.**

Proof: Using g to identify V and V^*, we obtain that the spaces $V^* \otimes V^*$ of bilinear symmetric forms and $\operatorname{End}(V) = V \otimes V^*$ are also identified. This identification is given by a map $A \mapsto g(A(\cdot), \cdot)$. By definition, the form $g_A(\cdot,\cdot) := g(A(\cdot),\cdot)$. is symmetric if and only if A is self-adjoint. ■

REMARK: This is just another way to construct the well-known **bijective correspondence between symmetric matrices and bilinear symmetric forms.**

REMARK: The same argument produces an **equivalence between Hermitian matrices $A^t = \overline{A}$ and Hermitian forms** on a complex vector space.

Normal form of a pair of bilinear symmetric forms

Theorem 1: (spectral theorem for self-adjoint operators)
Let A be a self-adjoint operator on a finite-dimensional space (V, g). **Then A can be diagonalized in an orthonormal basis.**

Theorem $1'$: ("principal axis theorem") Let $V = \mathbb{R}^n$, and $h, h' \in \operatorname{Sym}^2 V^*$ be two bilinear symmetric forms, with h positive definite. **Then there exists a basis $x_1, ..., x_n$ which is orthonormal with respect to h, and orthogonal with respect to h'.**

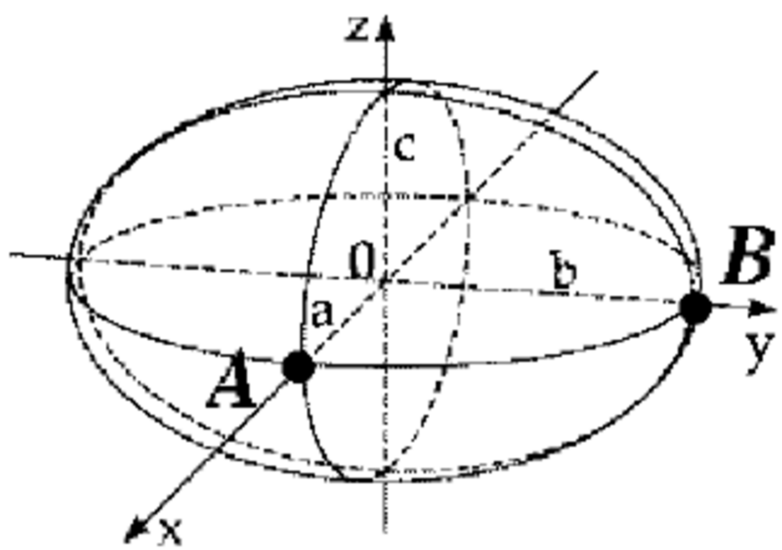
These theorems are clearly equivalent; I will give a proof later.

"Principal axis theorem"

REMARK: Theorem 1′ implies the following statement about ellipsoids: for any positive definite quadratic form q in $\mathbb{R}^n$, consider the ellipsoid

$$S = \{v \in V \mid q(v) = 1\}.$$

The group $SO(n)$ acts on $\mathbb{R}^n$ preserving the standard scalar product. **Then for some $g \in SO(n)$, $g(S)$ is given by equation $\sum a_i x_i^2 = 1$, where $a_i > 0$.** This is called **finding principal axes of an ellipsoid**.



Maximum of a quadratic form on a sphere

LEMMA: Let $V = \mathbb{R}^n$, and $h, h' \in \operatorname{Sym}^2 V^*$ be two bilinear symmetric forms, h positive definite, and $q(v) = h(v,v), q'(v) = h'(v,v)$ the corresponding quadratic forms. Consider q' as a function on a sphere $S = \{v \in V \mid q(v) = 1\}$, and let $x \in S$ be the point where q' attains maximum. Denote by $x^{\perp_h}$ and $x^{\perp_{h'}}$ the orthogonal complements with respect to h, h'. **Then $x^{\perp_h} = x^{\perp_{h'}}$.**

Proof: The tangent space $T_x S$ to a sphere S is $x^{\perp_h}$. Since $q'(x)$ reaches maximum on a sphere, one has $\frac{d}{d\varepsilon} q'(x + \varepsilon v) = 2h'(x, v) = 0$ for any $v \in T_x S = x^{\perp_h}$. This gives $h'(x, x^{\perp_h}) = 0$. ■

Theorem $1'$: ("principal axis theorem") Let $V = \mathbb{R}^n$, and $h, h' \in \operatorname{Sym}^2 V^*$ be two bilinear symmetric forms, with h positive definite. **Then there exists a basis $x_1, ..., x_n$ which is orthonormal with respect to h, and orthogonal with respect to h'.**

Proof: Let $q(v) = h(v,v), q'(v) = h'(v,v)$ the corresponding quadratic forms. Consider q' as a function on a sphere $S = \{v \in V \mid q(v) = 1\}$, and let $x_1 \in S$ be the point where q' attains maximum. Then $x_1^{\perp_h} = x_1^{\perp_{h'}}$. Using induction, we may assume that on $x_1^{\perp_h}$, Theorem 1 is already proven, and there exists a basis $x_2, ..., x_n$ orthonormal for h and orthogonal for h'. **Then $x_1, x_2, ..., x_n$ is orthonormal for h and orthogonal for h'.** ■

Weak convergence

DEFINITION: Let $x_i \in H$ be a sequence of points in a Hilbert space H. We say that x_i **weakly converges** to $x \in H$ if for any $z \in H$ one has $\lim_i g(x_i, z) = g(x, z)$.

REMARK: Let $y(i) = \alpha_j(i)e_j$ be a sequence of points in a a Hilbert space with orthonormal basis e_i. **Then $y(i)$ converges to $y = \sum_j \alpha_j e_j$ if and only if $\lim_i \alpha_j(i) = \alpha_i$.**

CLAIM: For any sequence $\{y(i) = \sum_j \alpha_j(i)e_j\}$ of points in a unit ball, **there exists a subsequence $\{\tilde{y}(i) = \tilde{\alpha}_j(i)e_i\}$ weakly converging to $y \in H$.**

Proof: Indeed, $|\alpha_j(i)| \leqslant 1$, hence there exist a subsequence $\tilde{y}(i) = \tilde{\alpha}_j(i)x_j$ with $\tilde{\alpha}_j(i)$ converging for each j. The limit belongs to the unit ball because otherwise $\left|\sum_{j=1}^n \tilde{\alpha}_j(i)e_j\right| > 1$, which is impossible. ■

REMARK: Note that **the function $x \longrightarrow |x|$ is not continuous in weak topology.** Indeed, weak limit of $\{e_i\}$ is 0. The proof above shows that $|\cdot|$ is semicontinuous.

Compact operators

DEFINITION: Precompact set is a set which has compact closure. **A compact operator** is an operator which maps bounded sets to precompact.

EXAMPLE: Let $A \in \operatorname{Hom}(H, H)$ be an operator on Hilbert spaces and $\{e_i\}$ an orthonormal basis in H. Let $A(e_i) = z_i$; assume that $\sum |z_i|^2 < \infty$. **Then A is compact.**

Proof. Step 1: Let $y(i) = \alpha_j(i) e_j$ be a sequence of points in a unit ball. Replacing $y(i)$ by a subsequence, we may assume that $y(i)$ weakly converges to y.

Step 2: Then

$$\lim_i A(y(i)) = \lim_i \lim_n A\left(\sum_{j=1}^n \alpha_j(i)\right) = \lim_n \sum_{j=1}^n \alpha_j A(e_i)$$

and this sequence converges in the usual topology on H, because α_j are bounded and $\sum_i |A(e_i)|^2$ is bounded. ■

Compact operators and weak convergence

THEOREM: Let $A: H \longrightarrow H$ be a compact operator. **Then A maps any weakly convergent sequence to a convergent one.**

Proof: Let $\{y_i\}$ be a sequence which weakly converges to y. Replacing $\{y_i\}$ by a subsequence, we may assume that $A(y_i)$ converges to z. Then $\lim_i g(A(y_i), v) = g(z, v)$ for any $v \in H$. However,

$$\lim_i g(A(y_i), v) = \lim_i g(y_i, A^*(v)) = g(y, A^*(v)) = g(A(y), v).$$

Then $g(z, v) = g(A(y), v)$ for all $v \in H$, giving $z = A(y)$. ■

REMARK: Converse is also true: **you can characterize a compact operator as one which maps weakly convergent sequences to convergent.** Indeed, unit ball is weakly compact, as we have shown, hence its image is precompact for any map which takes the weakly convergent sequences to convergent.

Spectral theorem

THEOREM: (Spectral theorem for self-adjoint operators)
Let $A: \ H \longrightarrow H$ be a compact self-adjoint operator on a Hilbert space. **Then A can be diagonalized** in a certain orthonormal basis $e_1, e_2, ...$, with $\lim_i \alpha_i = 0$.

Proof. Step 1: The eigenvalues converge to 0 because A is compact. Let $B \subset H$ be the unit ball, and X the closure of $A(B)$. Denote by $x \in X$ the vector where $|x|$ is maximal. We shall prove that $x = A(z)$. To finish the proof of Spectral Theorem **it would suffice to show that z is an eigenvector and $A(z^\perp) \subset z^\perp$.**

Step 2: Let $z_i \in B$ be a sequence such that $\lim_i A(z_i) = x$. Replacing z_i by a subsequence, we may assume that z_i weakly converges to $z \in B$. Then $A(z) = x$, because A maps weakly convergent sequences to convergent. **This implies that $x \in \operatorname{im} A$.**

Step 3: Let $z \in H$ be an element of the unit sphere such that $A(z) = x$. Then $|A(z)| = \|A\|$. Since A is self-adjoint, $g(A(z), A(z)) = g(A^2(z), z) = \|A\|^2$. This gives $\|A^2\| = \|A\|^2$, hence $|A^2(z)| = g(A^2(z), z)|$. Since $g(A^2(z), z) = |z||A^2(z)| \cos\varphi$, where φ is an angle between z and $A^2(z)$, the equality $g(A^2(z), z) = |z||A^2(z)|$ implies that z and $A^2(z)$ are proportional, hence x is an eigenvector for A^2.

Spectral theorem (2)

THEOREM: (Spectral theorem for self-adjoint operators)
Let $A: H \longrightarrow H$ be a compact self-adjoint operator on a Hilbert space. **Then A can be diagonalized** in a certain orthonormal basis $e_1, e_2, ...$, with $\lim_i \alpha_i = 0$.

Steps 2-3: We have shown that there exists a vector $z \in H$ in a unit sphere such that $|A(z)| = \|A\|$. Moreover, z is an eigenvector of A^2.

Step 4: Since A^2 is self-adjoint, for any space $W \subset H$ satisfying $A^2(W) \subset W$, one has $A^2(W^\perp) \subset W^\perp$. **We proved that A^2 is diagonal in an orthonormal basis.**

Step 5: This implies that H is an orthogonal direct sum of eigenspaces for A^2, which are finite-dimensional for non-zero eigenvalues, because A^2 is compact. Since A and A^2 commute, on each of these eigenspaces A acts as an adjoint operator, and we can apply the finite-dimensional spectral theorem. ■

REMARK: The same statement is true for Hermitian self-adjoint operators; the proof is the same.